The Root Locus Method

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The Root Locus (RL) method was developed by Evans [1] in 1948. The RL method allows to analyze and plot the roots of a polynomial with *real* coefficients that is *linear* in a single real parameter q. It may be written as

$$p(s,q) = p_0(s) + q\,p_1(s), \quad q \in [q^-;\, q^+]. \tag{1}$$

The classical application is a control system with open loop transfer function

$$k\,\frac{\mathrm{num}(s)}{\mathrm{den}(s)} = q\,\frac{p_1(s)}{p_0(s)} \tag{2}$$

with degree num $= m \leq$ degree den $= n$ and characteristic equation

$$1 + k\,\frac{\mathrm{num}(s)}{\mathrm{den}(s)} = 0, \quad k \in [0;\, +\infty).$$

The parameter q may also be a physical plant parameter entering *linearly* in the polynomial (1).

In MATLAB the root locus is computed by the command

$$\texttt{rlocus}(\texttt{p}_1,\ \texttt{p}_0,\ \texttt{q}),$$

where q is a user specified vector, e.g. q=0:0.01:2. The convenience of MATLAB has obstructed some useful analytical features of the RL that were well known to the precomputer generation of control engineers.

In the context of robust control some rules of the RL are important for the understanding of the parameter dependence of the roots of a polynomial. They are also useful in determining the necessary number of zeros of a controller. This is particularly important for the parameter space method which requires the assumption of a controller structure. The presentation of the rules follows Föllinger [2], where also proofs of the rules can be found.

Consider the polynomial (1) with

$$p_0(s) = \prod_{\nu=1}^{n} (s - s_\nu),$$

$$p_1(s) = \prod_{\mu=1}^{m} (s - s_\mu).$$

Rule 1:

The root locus is symmetric to the real axis and consists of n branches. For $q = 0$ they start at the roots s_ν of p_0 (poles of (2)), for $q \to +\infty$ m branches end at the roots s_μ of p_1 (zeros of (2)) and $n - m$ branches go to infinity (infinity branches).

Rule 2:

The asymptotes of the infinity branches intersect at the root center of gravity $(\sum_{\mu=1}^{m} s_\mu - \sum_{\nu=1}^{n} s_\nu)/(m - n)$. This rule can be remembered as follows: Assume an upwards unity force at each pole and a downwards unity force at each zero. Then the torques at the root center of gravity are balanced.

Rule 3:

The $n - m$ asymptotes form the angles

$$(2i + 1)\pi/(n - m), \quad i = 0, 1, \ldots n - m - 1$$

with the positive real axis.

The rules on the asymptotic behavior apply also for a polynomial dependency on q, e.g. $p(s) = p_0(s) + qp_1(s) + q^2 p_2(s)$. Now the roots of $p_2(s)$ have to be used as zeros.

Rule 4:

For small q the RL branches leave a double real pole under $\pm 90°$, a triple pole under $\pm 60°$ and $180°$, if there is an even number of real poles and zeros to the right of the triple pole, otherwise the angles are 0 and $\pm 120°$. A root locus branch leaves a complex pole under a direction such that the total phase angle is $-180° - i360°$, $i = 0, \pm 1, \pm 2, \ldots$. Under this angle the RL also enters the finite zeros for $q \to \infty$.

Rule 5:

A point on the real axis belongs to the RL if there is an odd number of real poles and zeros to the right of it.

Rule 6:

The RL branches off the real axis at branching points (the zeros have a multiplicity greater than 1). They satisfy $p(s) = 0$ and $p'(s) = 0$, that is the *resultant* of these two polynomials vanishes. Equivalently the zeros of the *discriminant* of $p(s)$ yield the values of q for the branching points. This formula can be simplified by taking $\ln p(s)$ first [2]. In the classical application of eq. (2) the branching points are determined by

$$\frac{d}{ds}\frac{\mathrm{num}(s)}{\mathrm{den}(s)} = \frac{\mathrm{num}'(s)\mathrm{den}(s) - \mathrm{num}(s)\mathrm{den}'(s)}{\mathrm{den}^2(s)} = 0.$$

There may be also multiple points in the complex plane, this corresponds to multiple complex zeros.

Rule 7:

The complex part of the RL for a system with one zero and two poles is a circle centered at the zero. For complex poles it runs through the poles, for two real poles on the same side of the zero it runs through the branching point in between. If the real poles are on opposite sides of the zero, then only a real RL occurs.

Remarks

1. If $q < 0$ we have "complementary" results. Replace $p_1(s)$ by $-p_1(s)$.

2. If $m \geq n$ use instead of the characteristic equation

$$1 + k\,\frac{\text{num}}{\text{den}} = 0$$

the equivalent equation

$$1 + \frac{1}{k}\,\frac{\text{den}}{\text{num}} = 1 + k^*\,\frac{\text{den}}{\text{num}} = 0.$$

3. A plot with different scaling on the coordinates axes yields incorrect informations on the angles.

4. For *nonlinear* dependency on q these rules are in general not applicable.

Some conclusions

A double integrator requires a negative real zero (rule 2 and 7), a triple integrator two zeros on the left side. The PID controller of exercise 4 must have a finite region of stability, because the asymptotes for k_D, k_P and k_I go into the right half plane. For the bus of exercise 10 we need at least two controller zeros (rule 4). For stability we need a minimum gain for returning the RL to the left half plane.

Examples

The characteristic polynomial of the crane is (see book, eq. 1.5.18 and example 3.5)

$$p(s) = a_0 s + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4$$

with

$$\begin{aligned} a_0 &= K k_1 g, \\ a_1 &= K k_2 g, \\ a_2 &= (m_L + m_C)g + K k_1 \ell - K k_3, \\ a_3 &= K k_2 \ell - K k_4, \\ a_4 &= \ell m_C \end{aligned}$$

(K a common feedback gain) and the nominal values are

$$\begin{aligned} g &= 10, \\ m_C &= 1000, \\ k_4 &= 0, \\ \\ K &= 1, \\ k_1 &= 500, \\ k_2 &= 2191, \\ k_3 &= -4299, \\ m_L &= 1500, \\ \ell &= 12. \end{aligned}$$

In the following examples we fix all but one parameter:

Case 1. The classical application:

$$1 + K \frac{k_2 l s^3 + (k_1 \ell - k_3)s^2 + 10 k_2 s + +10 k_1}{s^2(1000 \ell s^2 + 10 m_L + 10000)} = 0, \qquad 0 \leq K \leq 1.$$

Now let $K = 1$.

Case 2a, 2b, 2c. Controller design, synthesis:

$$1 + k_1 \frac{\ell s^2 + 10}{s[1000 \ell s^3 + k_2 \ell s^2 + (10 m_L + 10000 - k_3)s + 10 k_2]} = 0, \qquad 300 \leq k_1 \leq 700.$$

$$1 + k_2 \frac{s(\ell s^2 + 10)}{s[1000\ell s^3 + (10m_L + 10000 + k_1\ell - k_3)s + 10k_1]} = 0, \qquad 1691 \le k_2 \le 2691.$$

$$1 + k_3 \frac{-s^2}{1000\ell s^4 + k_2\ell s^3 + (10m_L + 10000 + k_1\ell)s^2 + 10k_2 s + 10k_1} = 0, \qquad -6299 \le k_3 \le -2299.$$

Case 3a, 3b. Analysis:

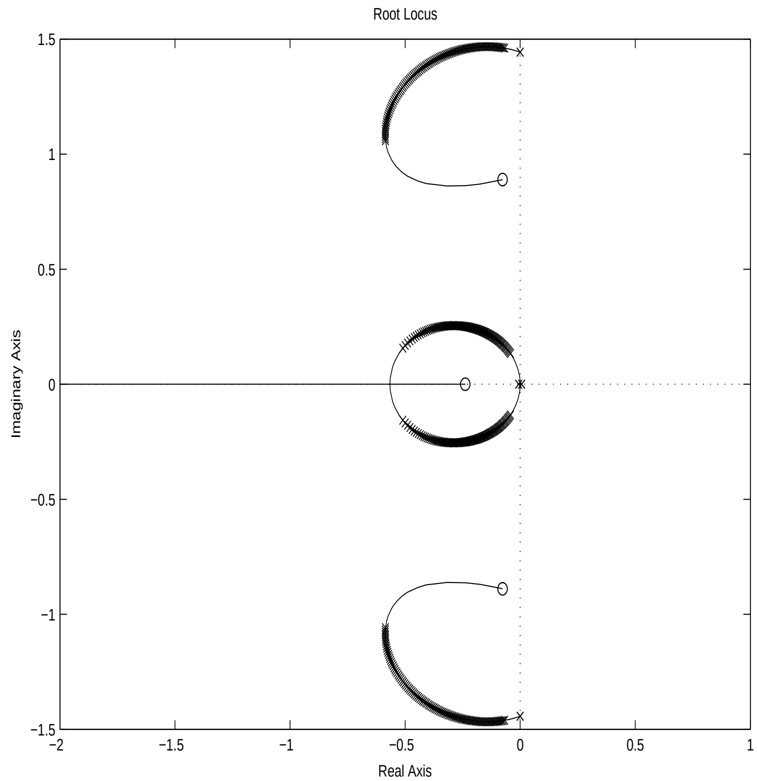
$$1 + m_L \frac{10s^2}{1000\ell s^4 + k_2\ell s^3 + (10000 + k_1\ell - k_3)s^2 + 10k_2 s + 10k_1} = 0, \qquad 1000 \le m_L \le 2000.$$

$$1 + \ell \frac{s^2(1000s^2 + k_2 s + k_1)}{(10m_L + 10000 - k_3)s^2 + 10k_2 s + 10k_1} = 0, \qquad 8 \le \ell \le 16.$$

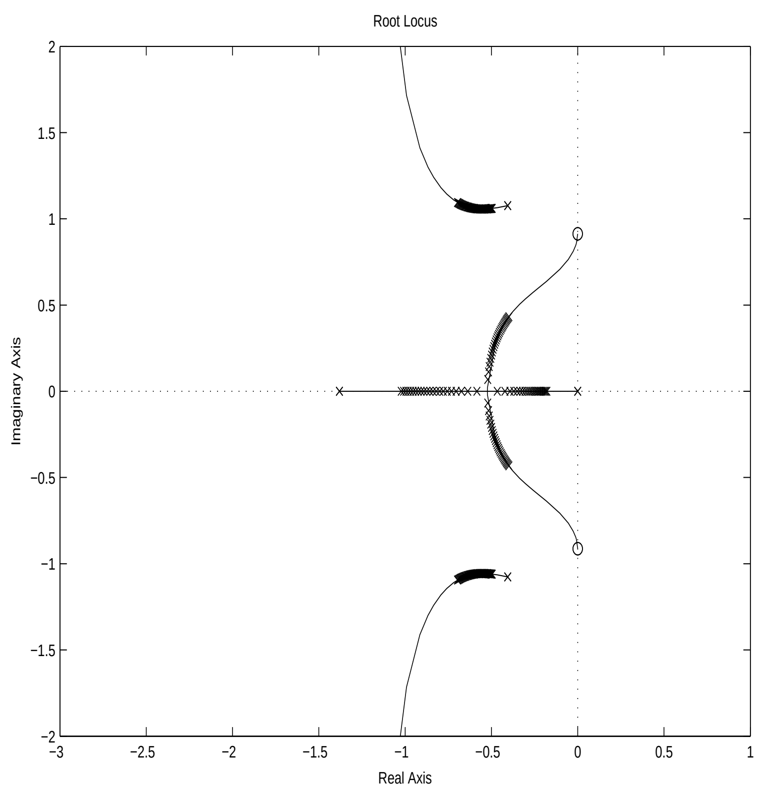
$m \ge n$!

(Cases 2c and 3a are essentially the same, k_3 and m_L enter in the same manner only in the coefficient a_2).

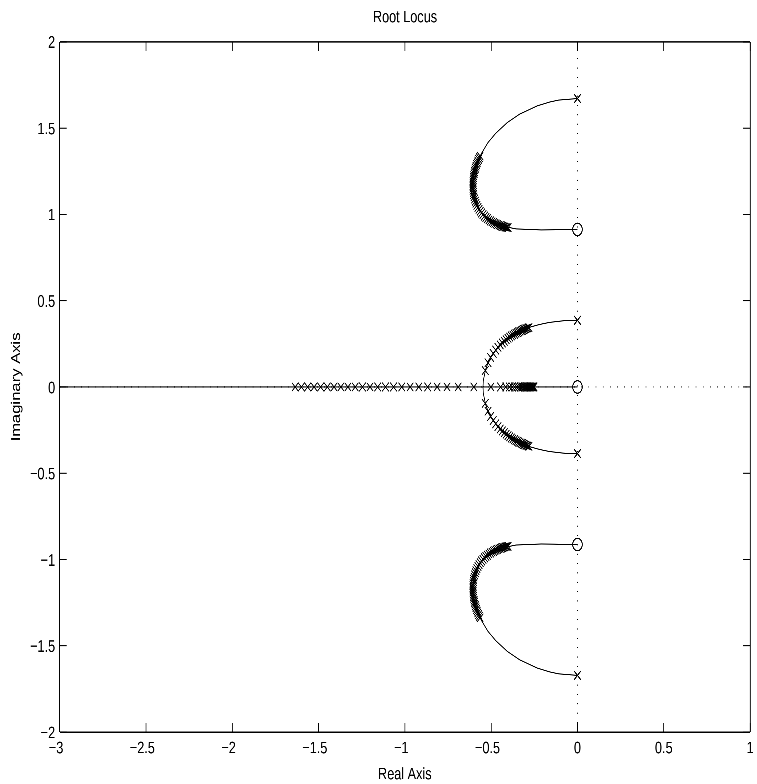
Case 1



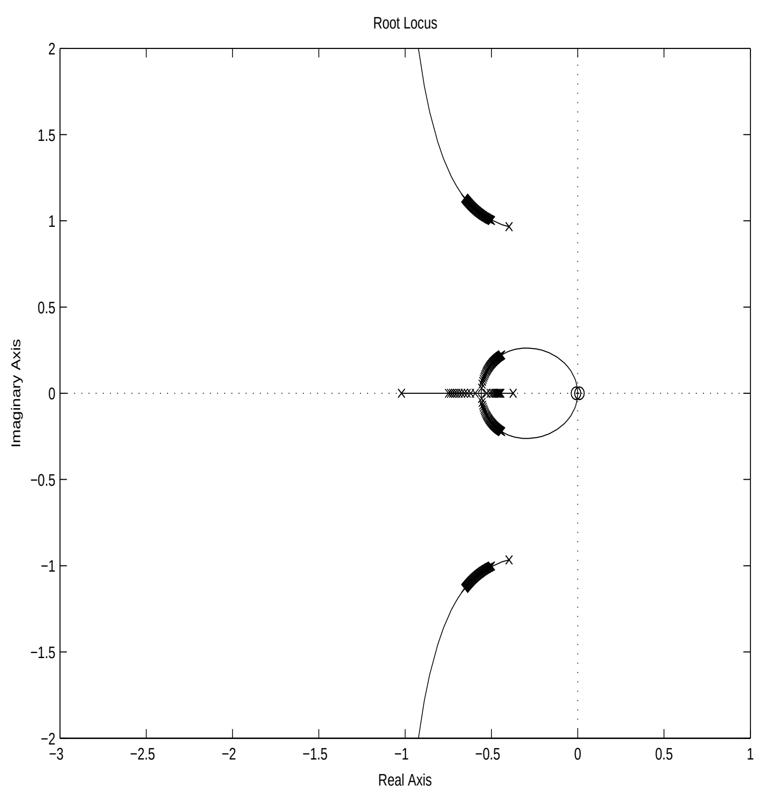
Case 2a



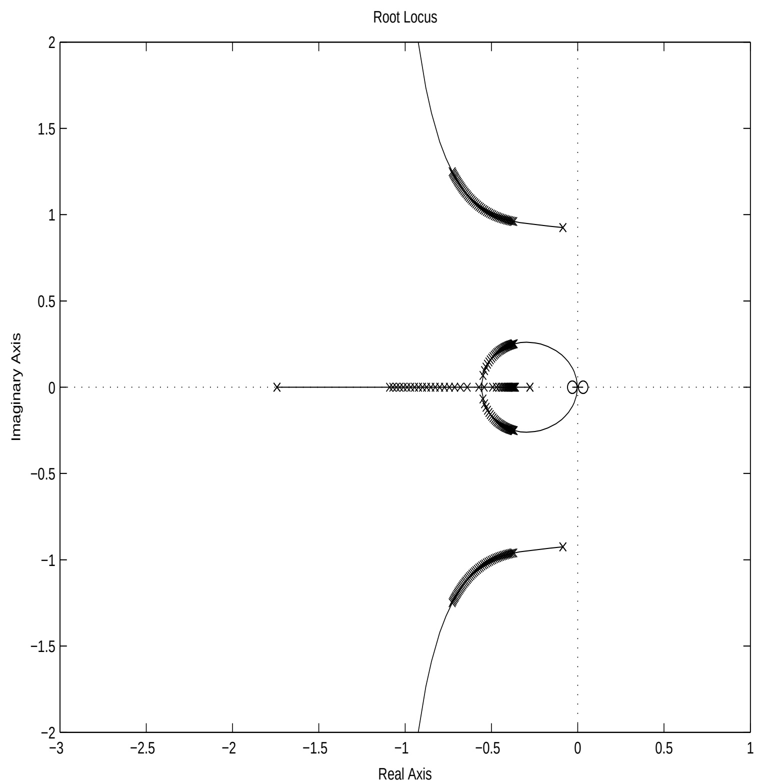
Case 2b



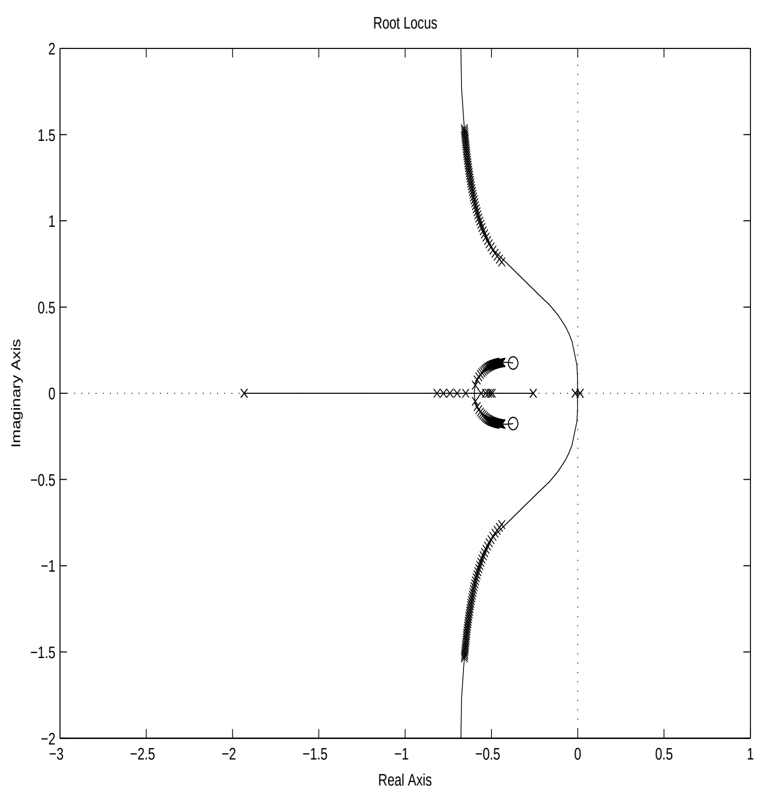
Case 2c



Case 3a

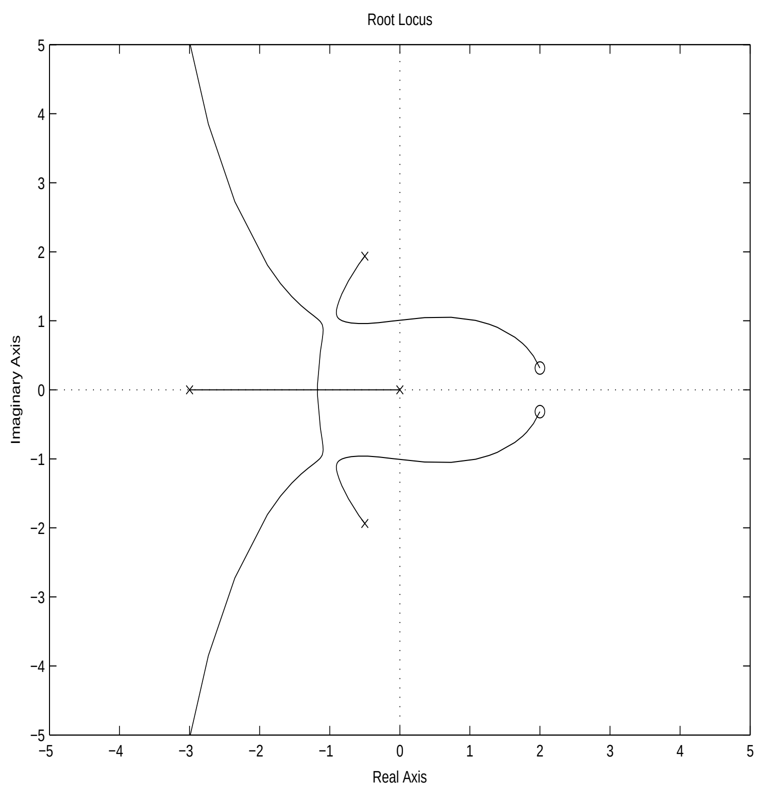


Case 3b



Example with complex branch points

$$G_1(s) = \frac{(s^2 - 4s + 4.1)}{s(s+3)(s^2+s+4)}$$

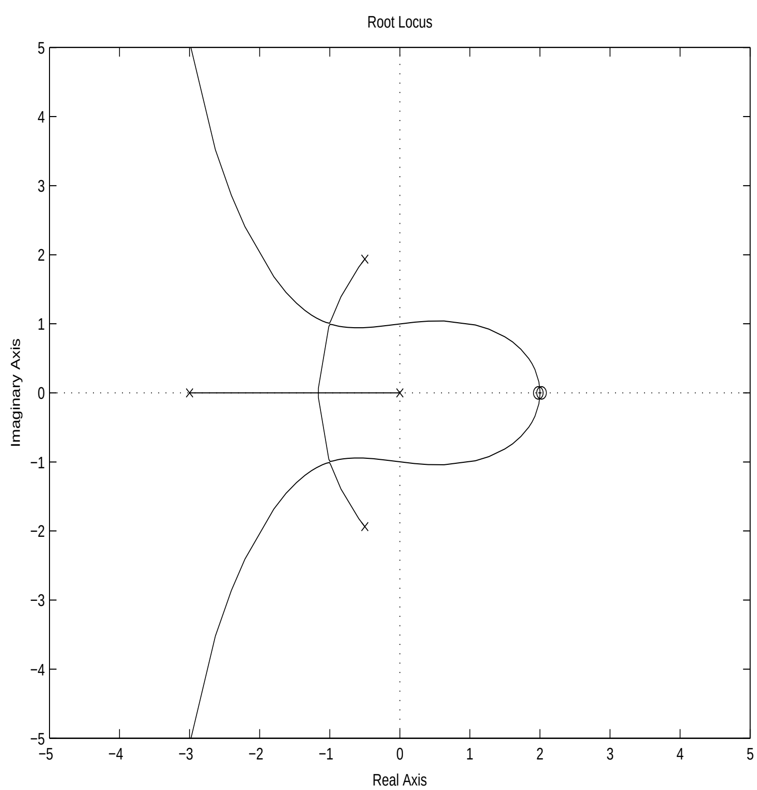


$$G_2(s) = \frac{(s-2)^2}{s(s+3)(s^2+s+4)}$$

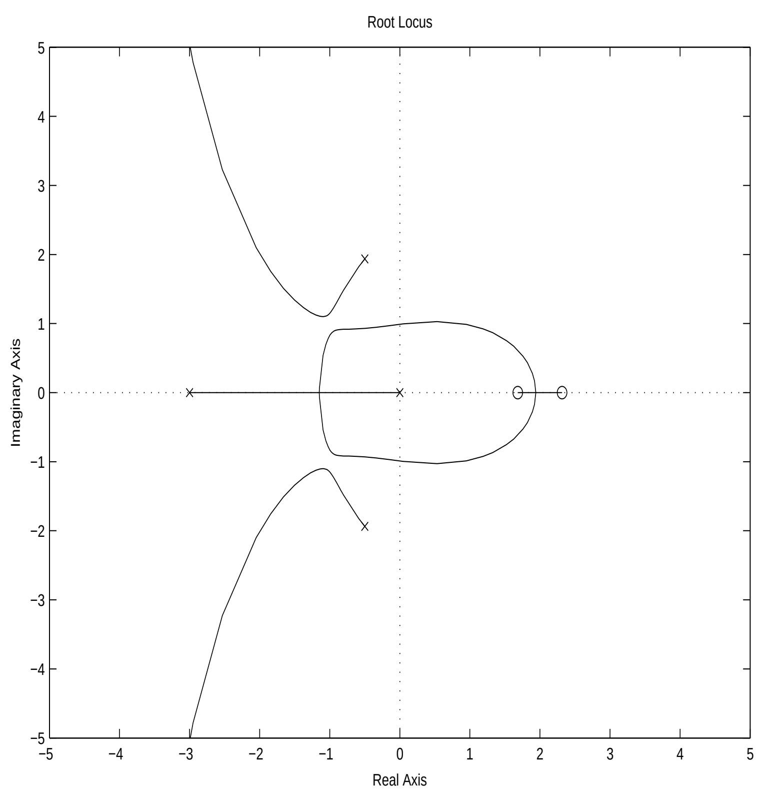
The discriminant of the polynomial $p(s) = s(s+3)(s^2+s+4) + k(s^2-4s+4)$ is

$$1600(k^2+150k-135)(k-1)^2.$$

We have for $k = -75 \pm 24\sqrt{10}$ the real break points at $s_{1/2} = 2 \mp \sqrt{10}$ and for $k = 1$ the complex break points at $s_{3/4} = -1 \pm \mathrm{j}$. One branch is part of a circle!



$$G_3(s) = \frac{(s^2 - 4s + 3.9)}{s(s+3)(s^2+s+4)}$$



References

[1] W.R.Evans, *Control-system Dynamics*, McGraw-Hill Book Company, Inc., New York, 1954.

[2] O.Föllinger, *Regelungstechnik*, 5.Auflage, Hüthig Verlag, Heidelberg, 1985.