J. Ackermann, Robust Control, WS 2010/11

Solution for Exam on 11 February, 2011

Problem 1

$p(s, q_1, q_2) = q_1 + (3 + q_2)s + 6s^2 + 4s^3 + s^4$

For which q_1, q_2 values do the roots have damping $> 1/\sqrt{2}$

Γ-stability boundary $\omega^2 = \sigma^2, \sigma < 0$

$$\begin{aligned}
\text{RRB} &: \; q_1 = 0 \;(\text{double root at } s = 0 \text{ for } q_1 = 0, q_2 = -3) \\
\text{IRB} &: \; a_4 = 1 \neq 0, \text{no IRB} \\
\text{CRB} &: \; d_0 = 1 \\
& \quad\; d_1 = 2\sigma \\
& \quad\; d_2 = 4\sigma^2 - 2\sigma^2 = 2\sigma^2 \\
& \quad\; d_3 = 4\sigma^3 - 4\sigma^3 = 0 \\
& \quad\; d_4 = -4\sigma^4
\end{aligned}$$

$$\begin{bmatrix} 1 & 2\sigma & 2\sigma^2 & 0 & -4\sigma^4 \\ 0 & 1 & 2\sigma & 2\sigma^2 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ 3 + q_2 \\ 6 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

1) $q_1 + 2\sigma(3 + q_2) + 12\sigma^2 - 4\sigma^4 = 0$

2) $3 + q_2 + 12\sigma + 8\sigma^2 = 0$

From 2) $\boxed{q_2 = -3 - 12\sigma - 8\sigma^2}$

into 1)

$q_1 + 2\sigma(-12\sigma - 8\sigma^2) + 12\sigma^2 - 4\sigma^4 = 0$

$$\boxed{q_1 = 4(3\sigma^2 + 4\sigma^3 + \sigma^4)}$$

Axis intersections

σ	q_1	q_2
0	0	-3
-0.317	0.737	0
-1	0	1
-1.183	-1.86	0
-3	0	-39

Extrema q_1

-0.634	1.392	1.392
-2.366	-19.39	-19.39

Extremum q_2

-0.75	1.266	1.5

Test point 1: $q_1 = 1, q_2 = 1$

$$\begin{aligned} p(s) &= 1 + 4s + 6s^2 + 4s^3 + s^4 \\ &= (s+1)^4 \qquad \Gamma - \text{stable} \end{aligned}$$

Test point 2: $q_1 = 1, q_2 = -50$

$$p(s) = 1 - 47s + 6s^2 + 4s^3 + s^4$$

unstable, $a_1 < 0$

The neighborhood of test point 1 is Γ-stable.

Problem 2

$$\begin{aligned} p(s) &= (K_I + K_P s + K_D s^2)(2-s) + s(2+s)(1+s)^2 \\ & \quad \text{RRB: } K_I = 0, \text{ no IRB} \\ A(s) &= 2 - s \\ R_A &= 2\ ,\ I_A = -\omega \\ B(s) &= 2s + 5s^2 + 4s^3 + s^4 \\ R_B &= -5\omega^2 + \omega^4 \\ I_B &= 2\omega - 4\omega^3 \\ K_P &= \frac{-\omega(-5\omega^2 + \omega^4) - 2(2\omega - 4\omega^3)}{\omega(4+\omega^2)} \\ &= \frac{-4 + 13\omega^2 - \omega^4}{4 + \omega^2} \\ K_P(\alpha) &= \frac{-4 + 13\alpha - \alpha^2}{4 + \alpha} \end{aligned}$$

Zeros: $\alpha^2 - 13\alpha + 4 = 0$

$\alpha_{1,2} = 6.5 \pm 6.1847$

$\alpha_1 = 0.3153\,,\ \alpha_2 = 12.6847$

Extrema:

$$\frac{dK_P}{d\alpha} = \frac{(13-2\alpha)(4+\alpha) - (-4+13\alpha-\alpha^2)}{(4+\alpha)^2}$$

$52 + 13\alpha - 8\alpha - 2\alpha^2 + 4 - 13\alpha + \alpha^2 = 0$

$56 - 8\alpha - \alpha^2 = 0$

$\alpha^2 + 8\alpha - 56 = 0$

$\alpha_{1,2} = -4 \pm 8.485$

$\alpha_1 = -12.485$, no real ω

$\alpha_2 = 4.485$

α	$K_P(\alpha)$
0	-1
0.3153	0
1	1.6
2	3
4	4
4.485	4.0294
5	4
8	3
12.6847	0
16	-2.6

see Fig. 2

K_P intervals:

$-\infty$	$<$	K_P	$<$	-1	2 boundary lines (incl. RRB)
-1	$<$	K_P	$<$	4.0294	3 boundary lines
4.0294	$<$	K_P	$<$	∞	1 boundary line

In the case of 1 or 2 boundary lines the candidate regions in (K_D, K_I)-plane are half planes or sectors that include infinity. They cannot be stable, however, because the root locus for the controller

$K_P + K\,\dfrac{\beta K_I + \gamma K_D s}{s}$, β, γ arbitrary

ends for $K \to \infty$ at the zero at $s = 2$.

If there exists a stabilizing interval for K_P, it must be contained in the interval $-1 < K_P < 4.0294$ with 3 boundary lines, i.e. it is a triangle.

Boundary lines result from $\omega = 0$ (RRB) and from singular frequencies at the roots of (2.6.5)

$$\begin{aligned} g(\omega) &= \omega K_P(R_A^2 + I_A^2) + R_A I_B - I_A R_B = 0 \\ &= \omega K_P(4+\omega^2) + 2(2\omega - 4\omega^3) + \omega(-5\omega^2 + \omega^4) = 0 \\ \frac{g(\omega)}{\omega} &= K_P(4+\omega^2) + 4 - 13\omega^2 + \omega^4 = 0 \end{aligned}$$

$\omega^2 = \alpha$

$\alpha^2 - (13 - K_P)\alpha + 4(1 + K_P) = 0$

$\alpha_{1,2} = 6.5 - 0.5K_P \pm \sqrt{(6.5 - 0.5K_P)^2 - 4 - 4K_P}$

The singular lines are by (2.6.8)

$$\begin{aligned} K_I &= \alpha_i K_D - [R_B(\omega_i) - I_A(\omega_i)\omega_i K_P]/R_A(\omega_i) \qquad i = 1,2 \\ &= \alpha_i K_D - [-5\alpha_i + \alpha_i^2 + \alpha_i K_P]/2 \\ &= \alpha_i[K_D - (\alpha_i + K_P - 5)/2] \end{aligned}$$

Example: $K_P = 1$

$$\begin{aligned} \alpha_{1,2} &= 6 \pm \sqrt{28} \; = 6 \pm 5.2915 \\ \alpha_1 &= 0.7085 \\ K_I &= 0.7085(K_D + 1.646) \\ \alpha_2 &= 11.2915 \\ K_I &= 11.29(K_D - 3.646) \end{aligned}$$

see Fig. 3

Test point $K_D = 1, K_I = 1$

$p(s) = 2 + 3s + 6s^2 + 3s^3 + s^4$

$a_i > 0 \qquad i = 0,1,2,3,4$

$$\Delta_3 = \begin{vmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 0 \\ 1 & 6 & 2 \\ 0 & 3 & 3 \end{vmatrix} = 27 > 0$$

The triangle of Fig. 3 is stable.

For $K_P \to -1$ the RRB and the CRB for $\omega \to 0$ become identical.

For $K_P \to 4.0294$ the two CRB's coincide. In both cases the stable region ends like the cutting edge of a knife.

Stabilizing values K_I, K_D exist for the range $K_P \in$(-1 ; 4.029).

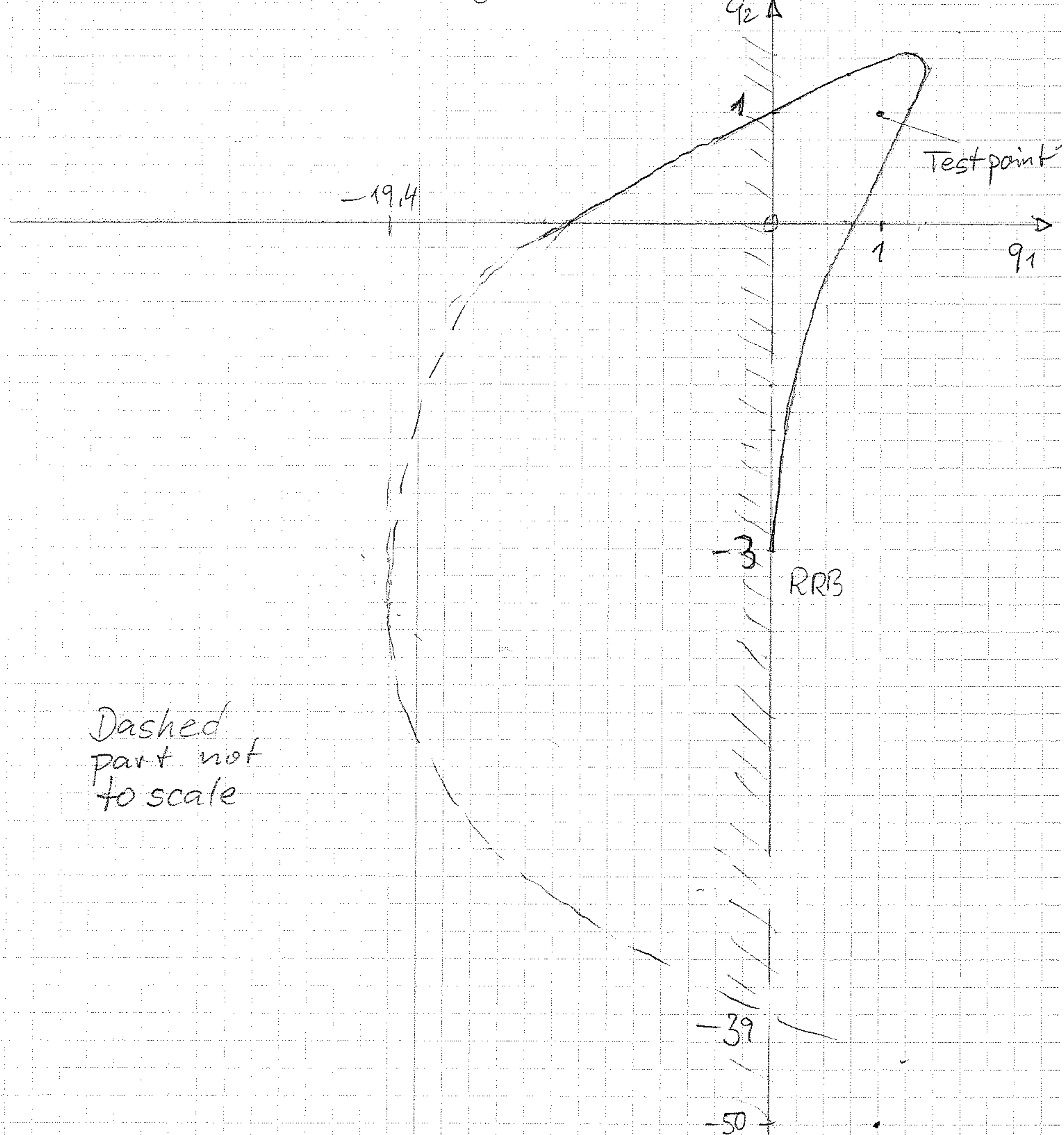
q_2
1
Testpoint
−19,4
0
1
q_1
−3
RRB
Dashed part not to scale
−39
−50
Testpoint 2

Fig. 1

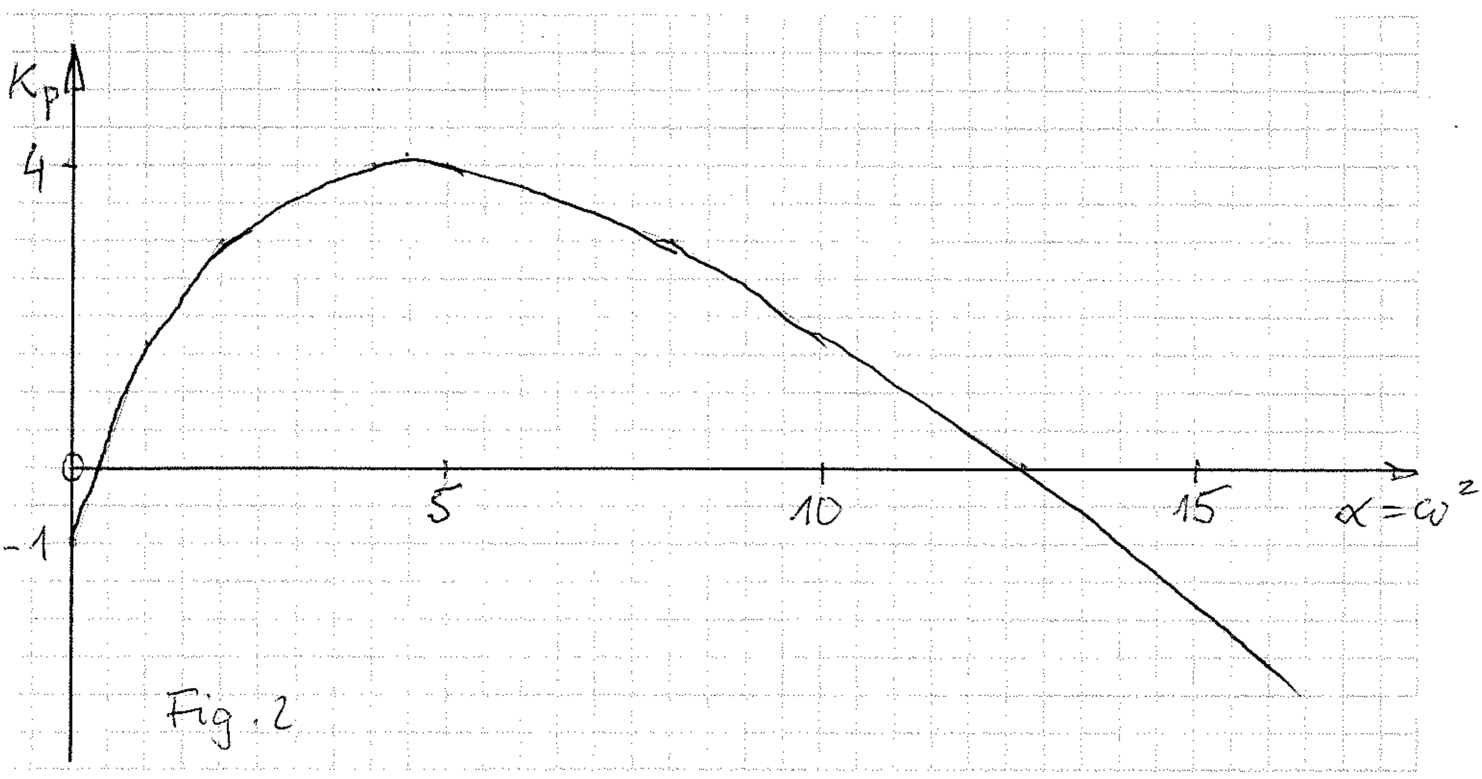

Fig. 2

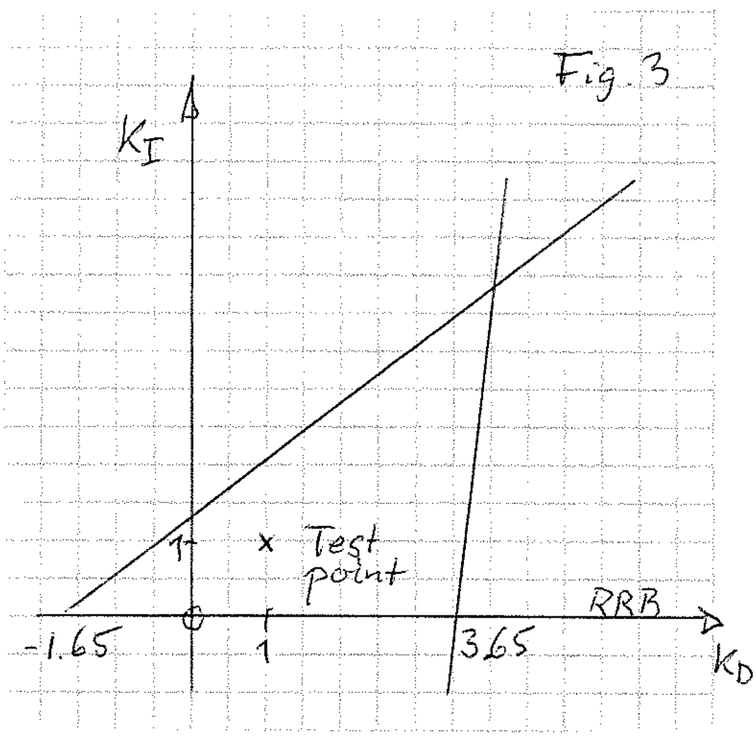

Fig. 3