External inputs

1) **Reference input** w for crab position x_1

$$u = k_1(w - x_1) - k_2 x_2 - k_3 x_3 - k_4 x_4 \qquad (1.5.15)$$

Transfer from

$$\boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{to} \quad \boldsymbol{x}(\infty) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

For a unit step input w an initial force peak $u(0) = k_1$ occurs.

Meet force limitation by a prefilter

$$w(s) = F(s)r(s), \text{ e.g. } F(s) = \frac{1}{1 + Ts}$$

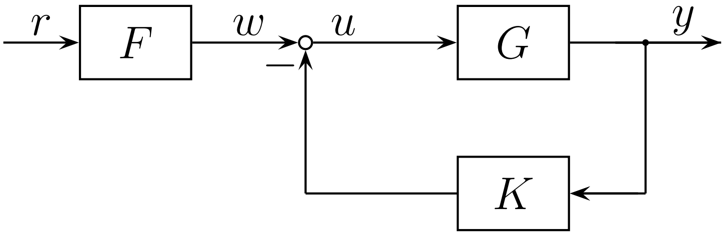

Figure 1.10. Control system with two degrees of freedom.

$$\begin{aligned} \boldsymbol{y} &= [x_1 \ x_3]^T \\ \boldsymbol{K}(s) &= \left[k_1 + \frac{k_2 s}{1 + Ts} \quad k_3 \right] \end{aligned}$$

Two degrees of freedom (Horowitz): Design K and F independently.

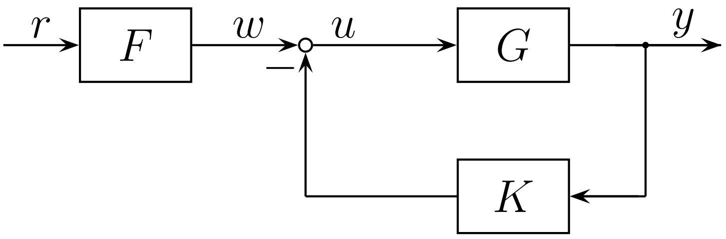

Figure 1.20. Control system with two degrees of freedom.

Closed-loop transfer function

$$G_{ry} = \frac{GF}{1 + GK} \tag{1.7.1}$$

Design goal: $G_{ry} = G_d$, **desired closed-loop transfer function**. Can be satisfied only if G_d meets three **feasibility constraints**:

i. G_{ry} must have at least the same relative degree as the plant G, otherwise F would not be realizable.
ii. G_{ry} must have at least the same dead time e^{-Ts} as the plant G, otherwise F would not be causal.
iii. G_{ry} must contain all zeros of G in the (closed) right half plane, because the non-minimum phase zeros (including zeros on the imaginary axis) cannot be cancelled.

Design procedure for nominal G

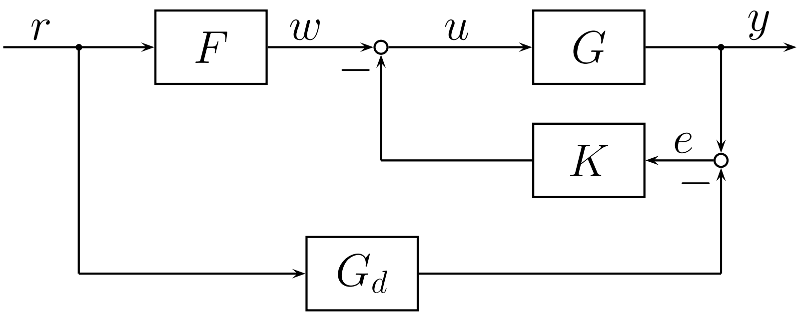

Figure 1.9. Modified two degree of freedom structure with reference model G_d

1. Specify a reference transfer function G_d that meets the feasibility constraints.
2. Choose the prefilter

$$F = G_d/G. \tag{1.7.2}$$

3. The ideal open-loop transfer function

$$G_{ry} = G_d = FG \tag{1.7.3}$$

would be obtained for $e = 0$. Then K has no influence on G_{ry}. K is designed to keep e small; this design step is independent of F and G_d and has to secure robust stability.

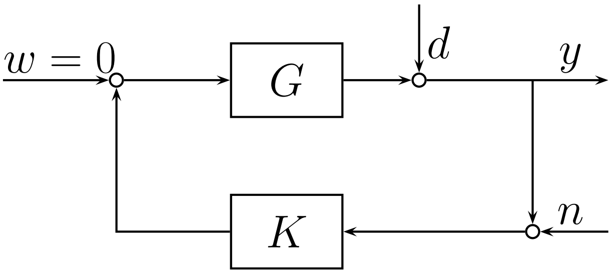

Figure 1.10. Plant disturbance d and sensor noise n

Disturbance rejection feedback structure

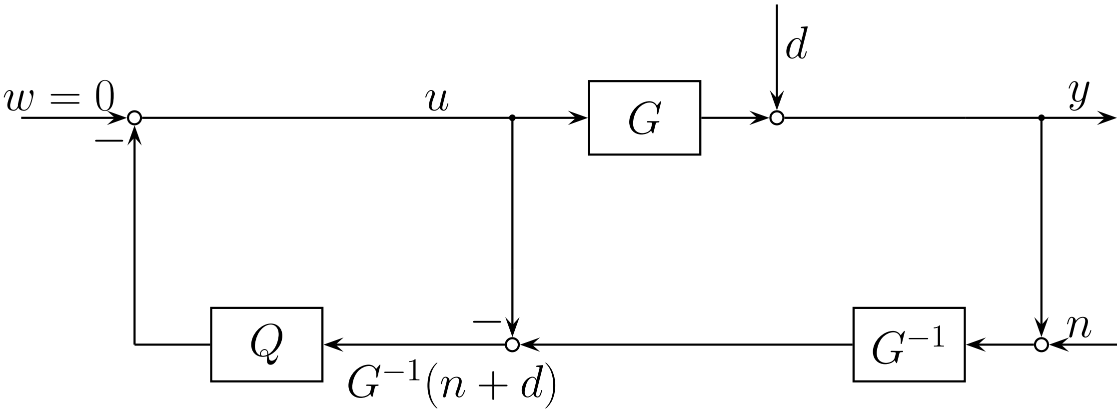

Figure 1.11. Disturbance rejection feedback structure

G^{-1} realizable in combination with a filter Q. Q must meet the feasibility constraints. Otherwise Q is free, it is chosen for a tradeoff between d and n.

$$y = (1 - Q)d - Qn. \tag{1.7.4}$$

Make $1 - Q$ small at low frequencies for d-rejection.

Make Q small at high frequencies for n-rejection.

→ Frequency domain specifications in Chapter 5.

For all inputs r, d and n

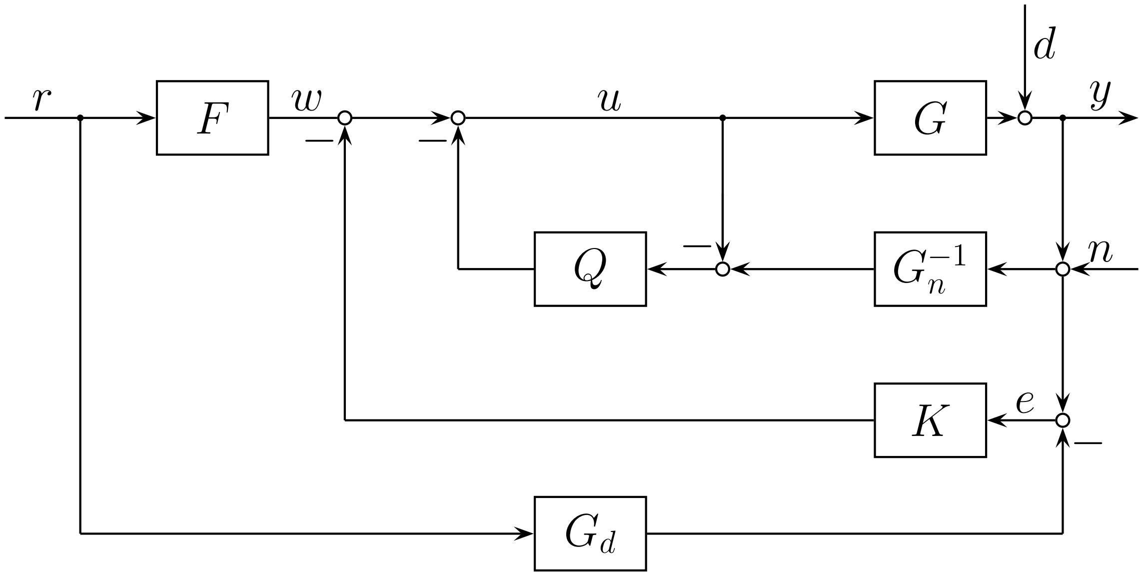

Figure 1.12. Control system structure for separation of design tasks.

- F for input r,
- Q for a tradeoff between inputs d and n,
- K for robust stabilization,
- G_d for model reference dynamics.

Unmodelled uncertainties at higher frequencies

Standard structures

$$\text{Additive perturbation } G = G_n + \Delta_m. \tag{1.7.5}$$

$$\text{Multiplicative perturbation } G = G_n(1 + \Delta_m). \tag{1.7.6}$$

$$\text{Feedback perturbation } G = \frac{G_n}{1 + G_n\Delta_m}. \tag{1.7.7}$$

Δ_m small in some frequency domain norm.

Frequency Loci Specification

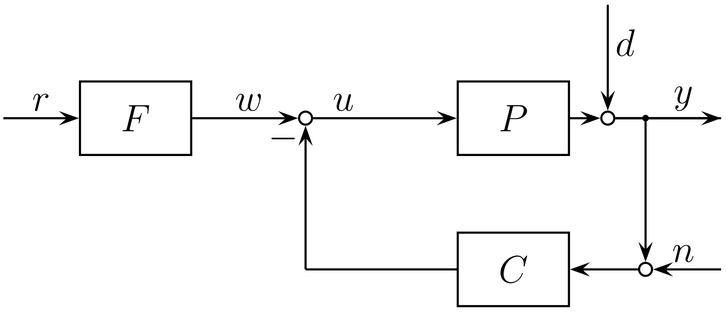

Figure 1.13. Control system with two degrees of freedom.

$$\begin{aligned}\boldsymbol{y} &= (\boldsymbol{I}+\boldsymbol{P}\boldsymbol{C})^{-1}(\boldsymbol{P}\boldsymbol{F}\boldsymbol{r}+\boldsymbol{d}-\boldsymbol{P}\boldsymbol{C}\boldsymbol{n}) = \boldsymbol{G}_{ry}\boldsymbol{r}+\boldsymbol{G}_{dy}\boldsymbol{d}+\boldsymbol{G}_{ny}\boldsymbol{n}\\ \boldsymbol{u} &= (\boldsymbol{I}+\boldsymbol{C}\boldsymbol{P})^{-1}(\boldsymbol{F}\boldsymbol{r}+\boldsymbol{C}\boldsymbol{d}-\boldsymbol{C}\boldsymbol{n}) = \boldsymbol{G}_{ru}\boldsymbol{r}+\boldsymbol{G}_{du}\boldsymbol{d}+\boldsymbol{G}_{nu}\boldsymbol{n}\end{aligned}$$

Γ-stability refers to the roots of

$$p(s,\boldsymbol{q},\boldsymbol{k}) = \det[\boldsymbol{I}+\boldsymbol{P}(s,\boldsymbol{q})\boldsymbol{C}(s,\boldsymbol{k})] = \det[\boldsymbol{I}+\boldsymbol{C}(s,\boldsymbol{k})\boldsymbol{P}(s,\boldsymbol{q})]$$

It does not consider the zeros of the six transfer functions.

Also frequency domain specifications are important, e.g. gain and phase margins.

Example 5.1

$$L(s,\boldsymbol{q}) = \frac{y(s)}{u(s)} = \frac{-q_1(5\,q_2\,s-8)}{5\,s^2+5(q_2+2)s+2} \qquad (5.1.1)$$

Unity feedback $u = w - y$

$$T(s,\boldsymbol{q}) = \frac{y(s)}{w(s)} = \frac{-q_1(5q_2\,s-8)}{5\,s^2+5(2+(1-q_1)q_2)\,s+2+8\,q_1} \quad (5.1.2)$$

Γ-stability

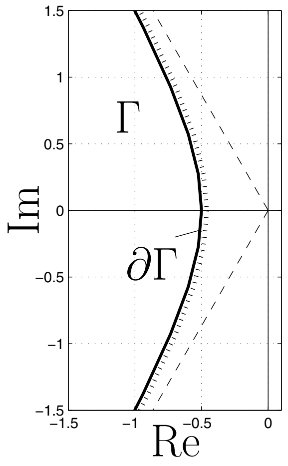

Figure 5.1. Definition of a Γ-region in the eigenvalue plane

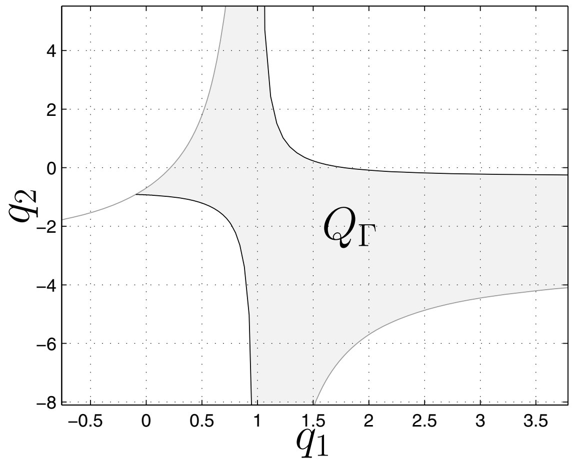

Figure 5.2. Mapped Γ-stable region Q_Γ of $G_{wy}(s,\boldsymbol{q})$ in the (q_1,q_2)-plane.

Let $q_1 = 1 \longrightarrow \Gamma$-stable for all q_2

$$den\,T(s,1,q_2) = 5s^2+10s+10$$

$$s_{1,2} = -1 \pm \mathrm{j}$$

Nyquist plots for $q_2 = 1$ and $q_2 = 10$

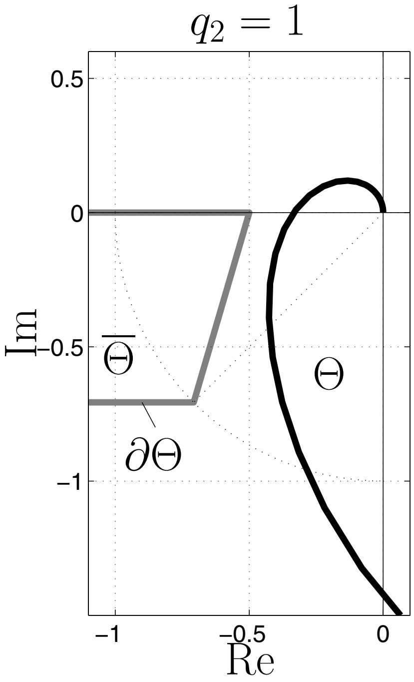

Figure 5.3. Nyquist plot for $q_1 = 1$, $q_2 = 1$ avoids the forbidden region $\overline{\Theta}$

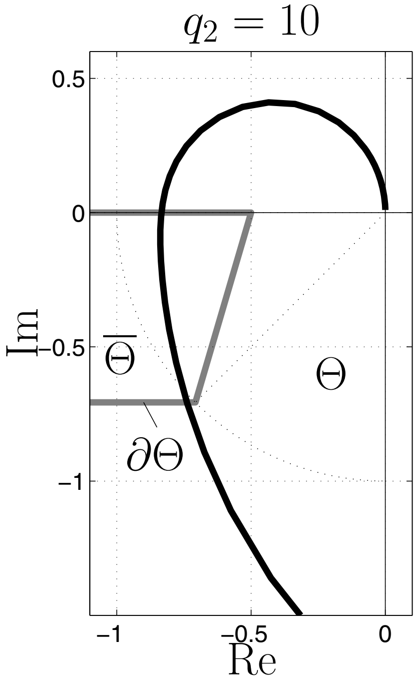

Figure 5.4. Nyquist plot for $q_1 = 1$, $q_2 = 10$ intersects the forbidden region $\overline{\Theta}$

$q_2 = 1$ has better gain and phase margin than $q_2 = 10$.

Define allowed Θ-region with 100% gain margin and 45^o phase margin.

Specification of Θ-stability: Nyquist plot must avoid forbidden $\overline{\Theta}$-region.

$q_1 = 1$ is Θ-stable, $q_2 = 10$ is not $\Theta-$stable.

Map $\partial\Theta$ to (q_1, q_2)-plane

Describing functions and dual locus plot

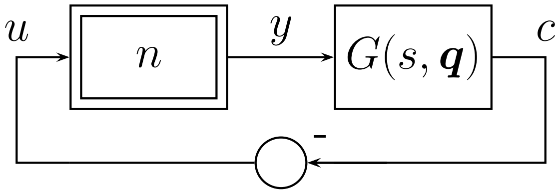

Figure 5.10. Single nonlinear loop with uncertain linear part $G(s, \boldsymbol{q})$.

$G(\mathrm{j}\omega, \boldsymbol{q})$ has low-pass property for all $\boldsymbol{q} \in Q$.

For a **limit cycle** all signals are approximated by the basic sinusoidal term of a Fourier series expansion. Assume

$$u = A \sin \omega t$$

then

$$\begin{aligned} y &= N(A, \omega) \sin[\omega t - \varphi(A, \omega)] \\ &+ \text{neglected terms of frequencies } 2\omega, 3\omega, 4\omega \ldots \end{aligned}$$

Describing function: $n = N(A, \omega) e^{-\mathrm{j}\varphi(A,\omega)}$

For static nonlinearities: $n = N(A)$

Characteristic equation

$1 + nG(\mathrm{j}\omega, \boldsymbol{q}) = 0$

Dual locus plot

$G(\mathrm{j}\omega, \boldsymbol{q}) = -1/n$

Example: Saturation

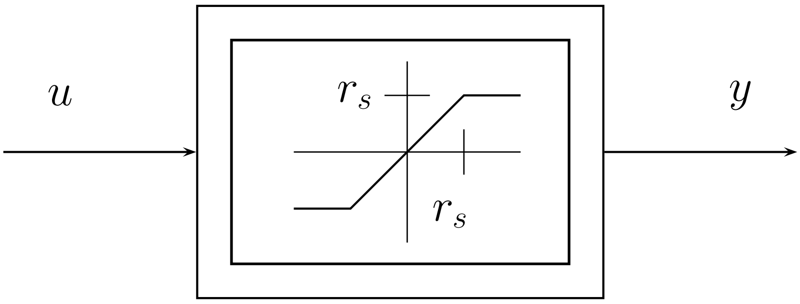

Figure 5.13. Saturation.

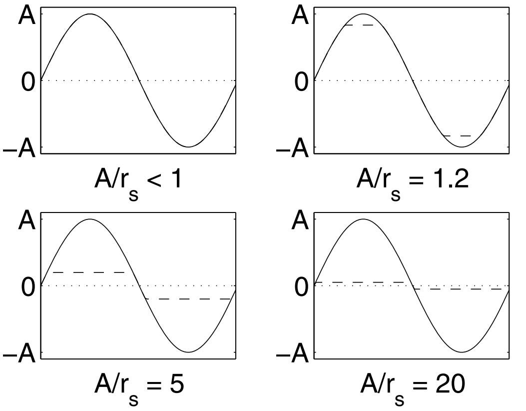

Figure 5.14. Saturation output (dotted) for sinusoidal input.

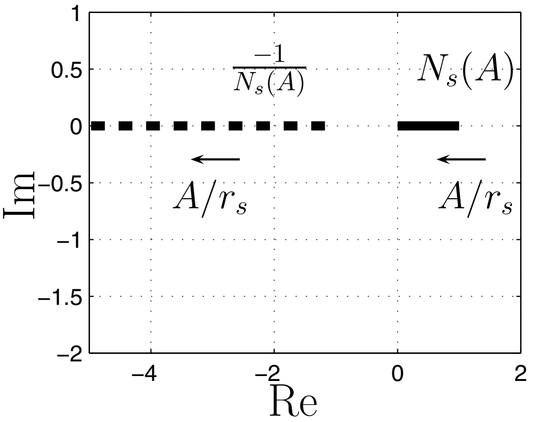

Figure 5.15. Describing function locus $N_s(A)$ and its negative inverse $-1/N_s(A)$.

Test:

- For one $\boldsymbol{q} \in Q$ no limit cycles
- family of Nyquist plots $G(\mathrm{j}\omega, \boldsymbol{q})$, $\boldsymbol{q} \in Q$ must avoid $-1/N_S(A)$.

Example: Rate saturation

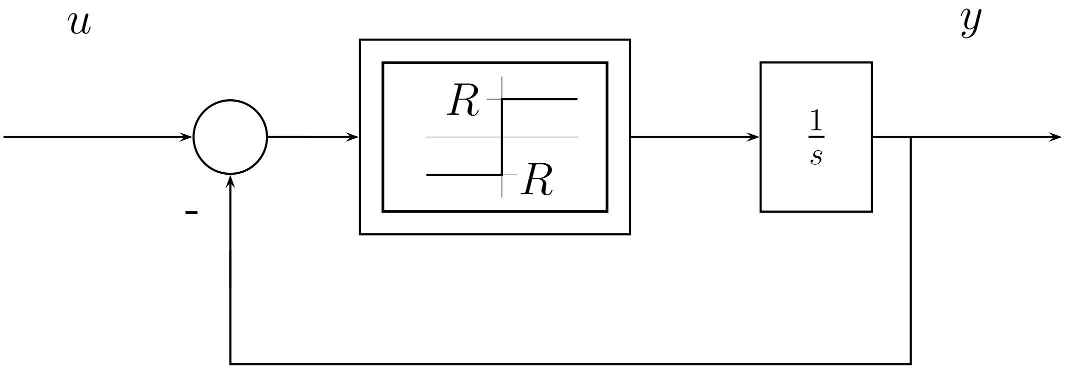

Figure 5.16. Block diagram of rate saturation.

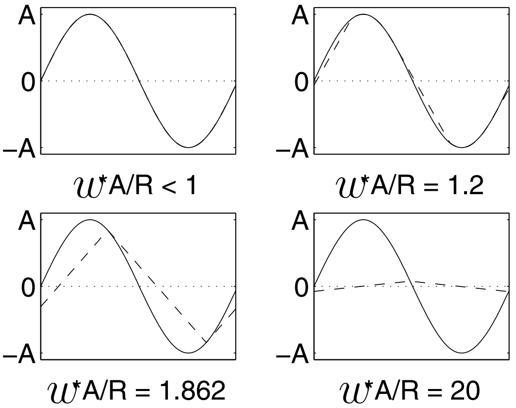

Figure 5.17. Rate saturated output (dotted) for sinusoidal input.

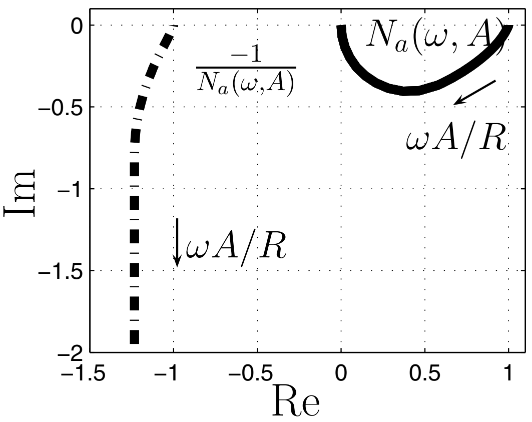

Figure 5.18. Describing function locus of a rate saturation and its negative inverse.

Test:

- For one $\boldsymbol{q} \in Q$ no limit cycles
- family of Nyquist plots $G(\mathrm{j}\omega, \boldsymbol{q})$, $\boldsymbol{q} \in Q$ must avoid $-1/N(A,\omega)e^{-\mathrm{j}\varphi(A,\omega)}$.

Popov Criterion

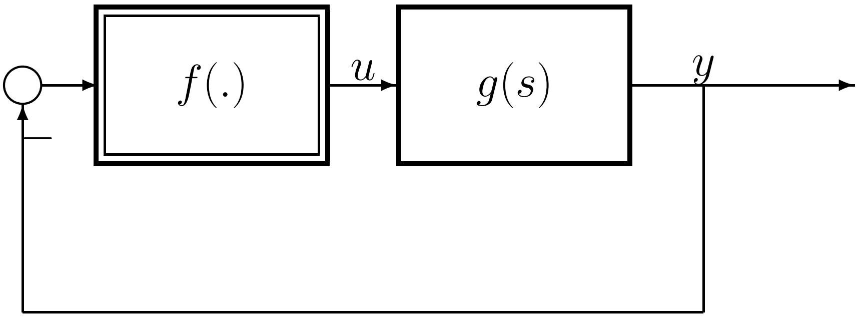

Figure 5.5. Nonlinear function in the feedback loop.

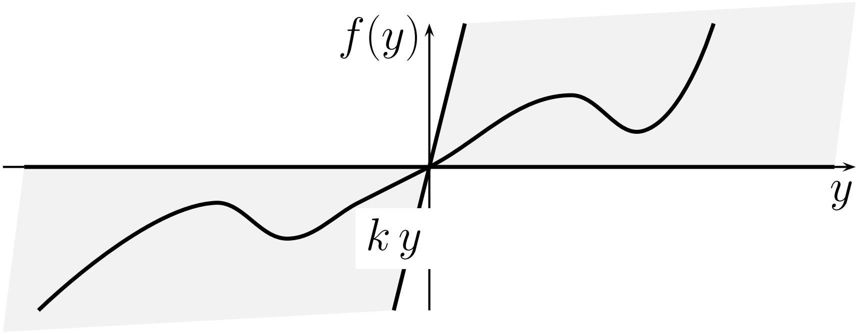

Figure 5.6. Sector of nonlinearity.

Sufficient stability condition:
Stable for all piecewise continuous nonlinearities in the sector $[0; k]$ if Popov plot

$$g_p(\mathrm{j}\omega) := \mathrm{Re}\ g(\mathrm{j}\omega) + \mathrm{j}\omega\ \mathrm{Im}\ g(\mathrm{j}\omega),\ \ \omega \geq 0 \tag{5.1.7}$$

entirely to the right of a Popov line with real axis intersection at $-1/k$ and arbitrary slope.

Largest Popov sector from tangent Popov line

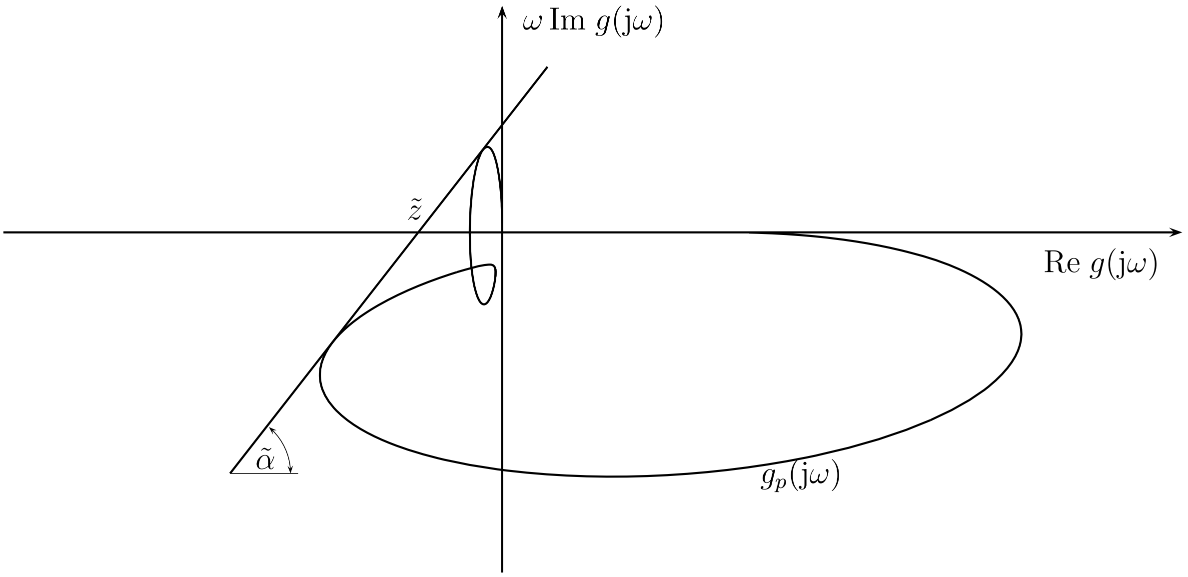

Figure 5.7. Absolute stability and Popov sector.

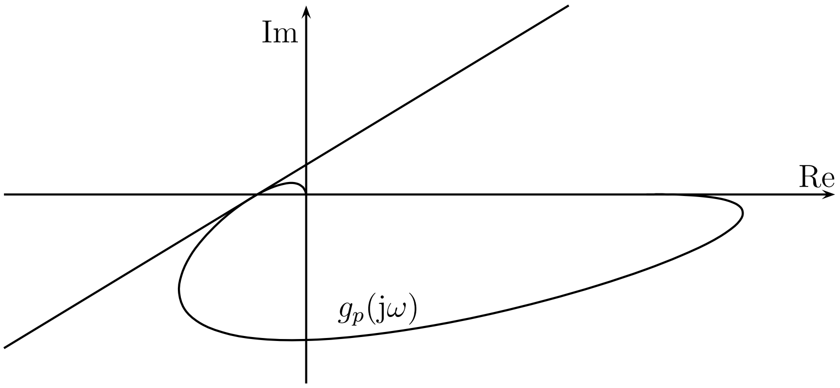

Figure 5.8. An example for identical Nyquist and Popov sectors.

If Nyquist and Popov sectors are identical, then the stability condition is necessary and sufficient.

A family of Popov plots $g_p(\mathrm{j}\omega, \boldsymbol{q}), \boldsymbol{q} \in Q$ results in a family of tangent Popov lines. The most left real axis intersection of all tangent Popov lines yields $-1/k^+$ (i.e. the maximal common Popov sector $[0, k^+]$).

Note that this k^+ is larger or equal to the k resulting from a common Popov line for the family $g_p(\mathrm{j}\omega)$, $\boldsymbol{q} \in Q$.

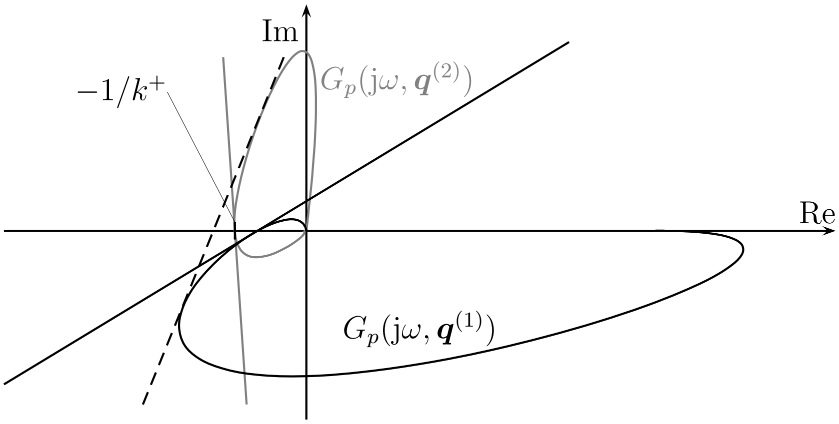

Figure 5.9. Different Popov-lines apply to different members of a Popov plot family

5.2 Mapping of Frequency Domain Margins

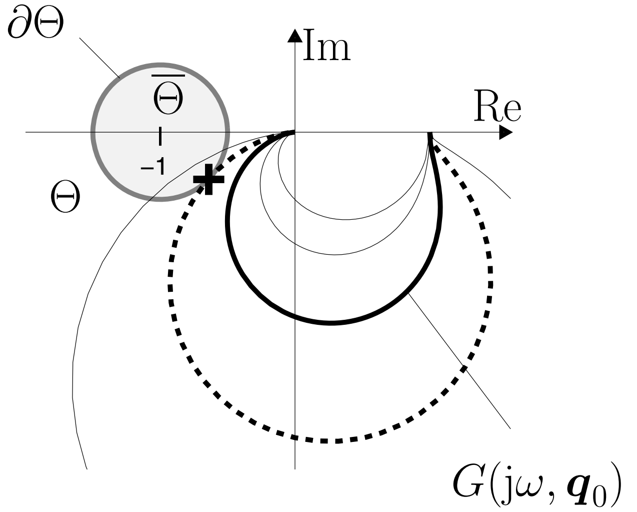

Figure 5.19. Example for a Θ-stability definition in the Nyquist diagram

$$G(\mathrm{j}\omega, \boldsymbol{q}) \;=\; R_G(\omega, \boldsymbol{q}) + \mathrm{j}I_G(\omega, \boldsymbol{q}) \tag{5.2.3}$$

$$\partial\Theta = \{x + \mathrm{j}y \mid F_{\partial\Theta}(x, y) = 0\} \tag{5.1.17}$$

Start from a $\Theta-$stable $\boldsymbol{q}_0$.

For which $\boldsymbol{q}$ does $G(\mathrm{j}\omega, \boldsymbol{q})$ change from $\Theta-$stable to $\Theta-$unstable?

a) *Tangent condition*

$$F_{\partial\Theta}(R_G(\omega, \boldsymbol{q}), I_G(\omega, \boldsymbol{q})) = 0 \tag{5.2.12}$$

$$\frac{\partial}{\partial\omega} F_{\partial\Theta}(R_G(\omega, \boldsymbol{q}), I_G(\omega, \boldsymbol{q})) = 0 \tag{5.2.13}$$

b) *Point condition* at $G = z^*$

$$Q_{z^*} = \left\{\ \boldsymbol{q} \in \mathbb{R}^n \mid G(\mathrm{j}\omega, \boldsymbol{q}) = z^*, \quad \omega \in \mathbb{R}_0^+\ \right\}, \quad (5.2.4)$$

$$p(\mathrm{j}\omega, \boldsymbol{q}, z^*) = N_G(\mathrm{j}\omega, \boldsymbol{q}) - z^* D_G(\mathrm{j}\omega, \boldsymbol{q}) = 0 \qquad (5.2.5)$$

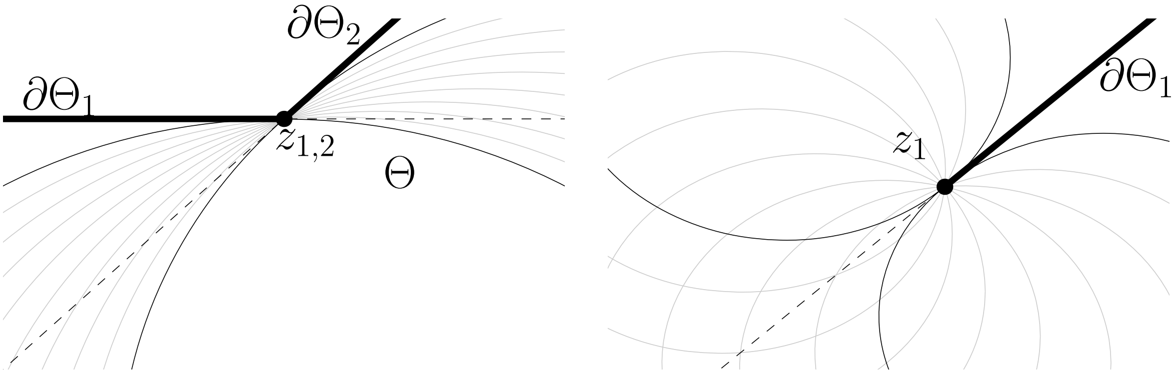

Figure 5.20. Application of point and tangent conditions to compound (l.h.s.) and closing (r.h.s.) Θ-boundaries.

c) *Frequency endpoint condition*

for $\omega = 0$ or $\omega = \infty$

d) *Infinite locus magnitude condition*

for pole of $G(s, \boldsymbol{q})$ on the imaginary axis.

$G(\mathrm{j}\omega, \boldsymbol{q})$ jumps from Θ to $\overline{\Theta}$ via infinity.

Example

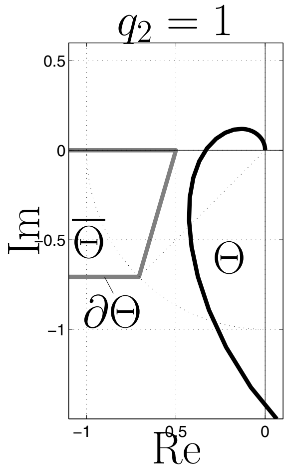

Figure 5.3. Nyquist plot for $q_1 = 1$, $q_2 = 1$ avoids the forbidden region $\overline{\Theta}$

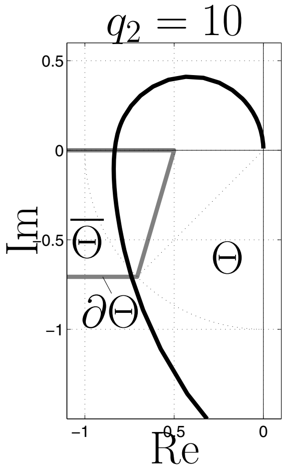

Figure 5.4. Nyquist plot for $q_1 = 1$, $q_2 = 10$ intersects the forbidden region $\overline{\Theta}$

a) Tangent condition for slanted line

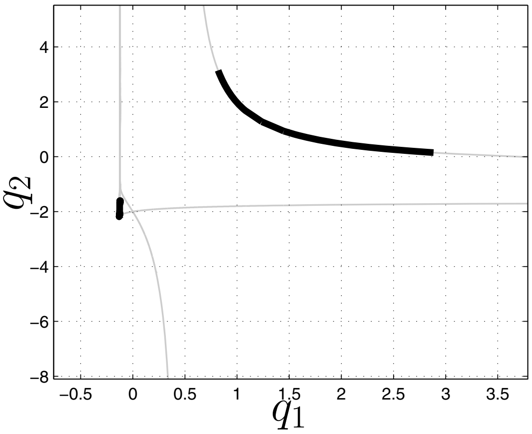

Figure 5.24. Tangent condition

grey: entire straight line

black: boundary segment

Similar for two horizontal line segments

b) Point condition for $z^* = -1/2$ and $z^* = e^{\mathrm{j}\cdot 5/4\pi}$

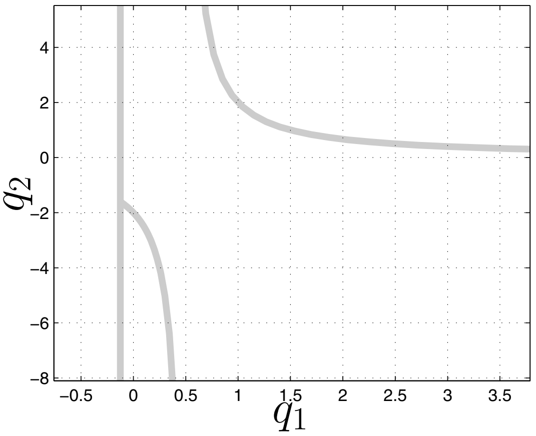

Figure 5.21. Point Condition for $z^* = -1/2$

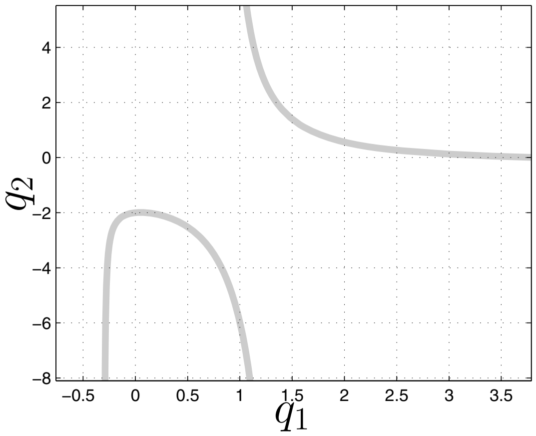

Figure 5.22. Point Condition for $z = e^{\mathrm{j}\cdot 5/4\pi}$

Overlay with Γ−stable region

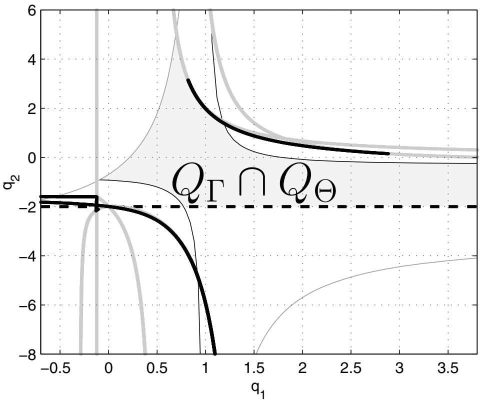

Figure 5.25. Boundaries and stability region $Q_\Gamma \cap Q_\Theta$ for simultaneous Γ- and Θ-stability

c) Frequency endpoint
$\omega = 0$ at $z^* = -1/2 \rightarrow q_1 = -1/8$
contained in point condition

d) Infinite magnitude

$$G_{uy}(s, \boldsymbol{q}) = \frac{y(s)}{u(s)} = \frac{-q_1(5\,q_2\,s - 8)}{5\,s^2 + 5(q_2 + 2)s + 2} \qquad (5.1.1)$$

$q_2 = -2 \rightarrow$ poles on imaginary axis.

Example 5.2

Crane with $m_C = 1000$ and feedback

$u = -[500\ 2856\ \ -22800\ 0]\boldsymbol{x}$

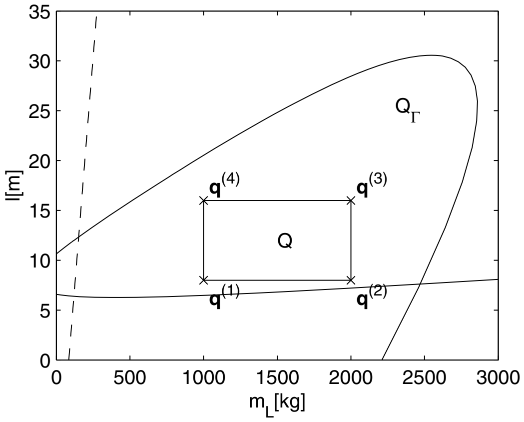

Figure 3.16. Robustness analysis in the (m_L, ℓ)-plane.

Further restriction by saturation

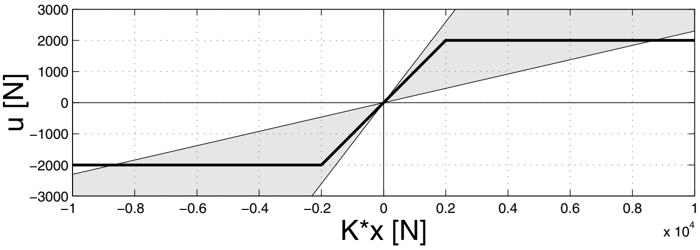

Figure 5.26. Saturation of the crane input (i.e. crab acceleration)

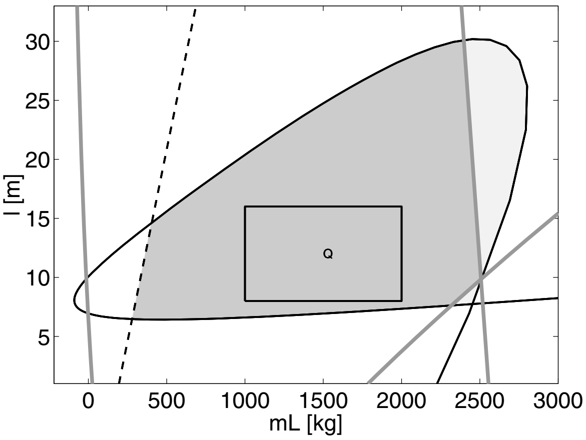

Figure 5.27. Stability region of the crane for simultaneous Γ- and Θ-stability (compare to Fig. 3.16)

$\boldsymbol{\mathcal{B}}$-Stability

Specification on frequency response magnitude

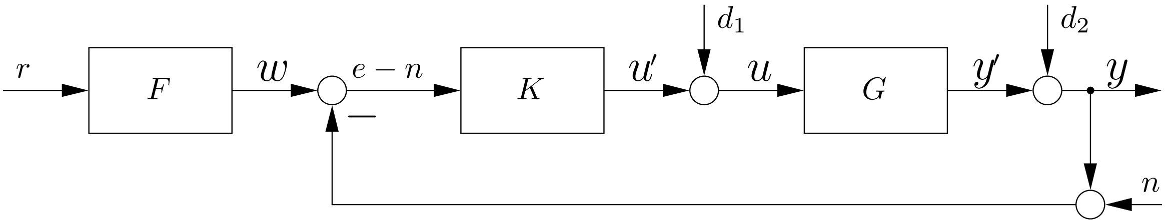

Figure 5.30. Standard single-loop feedback system.

$$\underbrace{\begin{bmatrix} e \\ u \\ y \end{bmatrix}}_{\boldsymbol{z}} = \underbrace{\begin{bmatrix} S & T & -S_G & -S \\ S_K & -S_K & S & -S_K \\ T & -T & S_G & S \end{bmatrix}}_{\boldsymbol{R}} \cdot \underbrace{\begin{bmatrix} w \\ n \\ d_1 \\ d_2 \end{bmatrix}}_{\boldsymbol{w}} \tag{5.3.1}$$

$$S = \frac{1}{1 + K\,G} \tag{5.3.2}$$

$$T = \frac{KG}{1 + K\,G} = K\,G\,S = 1 - S \tag{5.3.3}$$

$$S_K = \frac{K}{1 + K\,G} = K\,S \tag{5.3.4}$$

$$S_G = \frac{G}{1 + K\,G} = G\,S \tag{5.3.5}$$

1) *Tracking* in a frequency range $[0; \omega_S]$

$$|S(j\omega)| \ll 1 \quad \text{for} \quad 0 \leq \omega \leq \omega_S \; . \tag{5.3.9}$$

2) *Disturbance attenuation* in a frequency range $[0; \omega_S]$

$$|S_G(j\omega)| \ll 1 \quad \text{for} \quad 0 \leq \omega \leq \omega_S \; . \tag{5.3.10}$$

3) *Noise rejection* in a frequency range $[\omega_T; \infty]$

$$|T(j\omega)| \ll 1 \quad \text{for} \quad \omega_T \leq \omega < \infty \; . \tag{5.3.11}$$

4) *Control input limitation* within actuator bandwidth $[0; \omega_{act}]$

$$|S_K(\mathrm{j}\omega)| \text{ small for } 0 \leq \omega \leq \omega_{act}$$

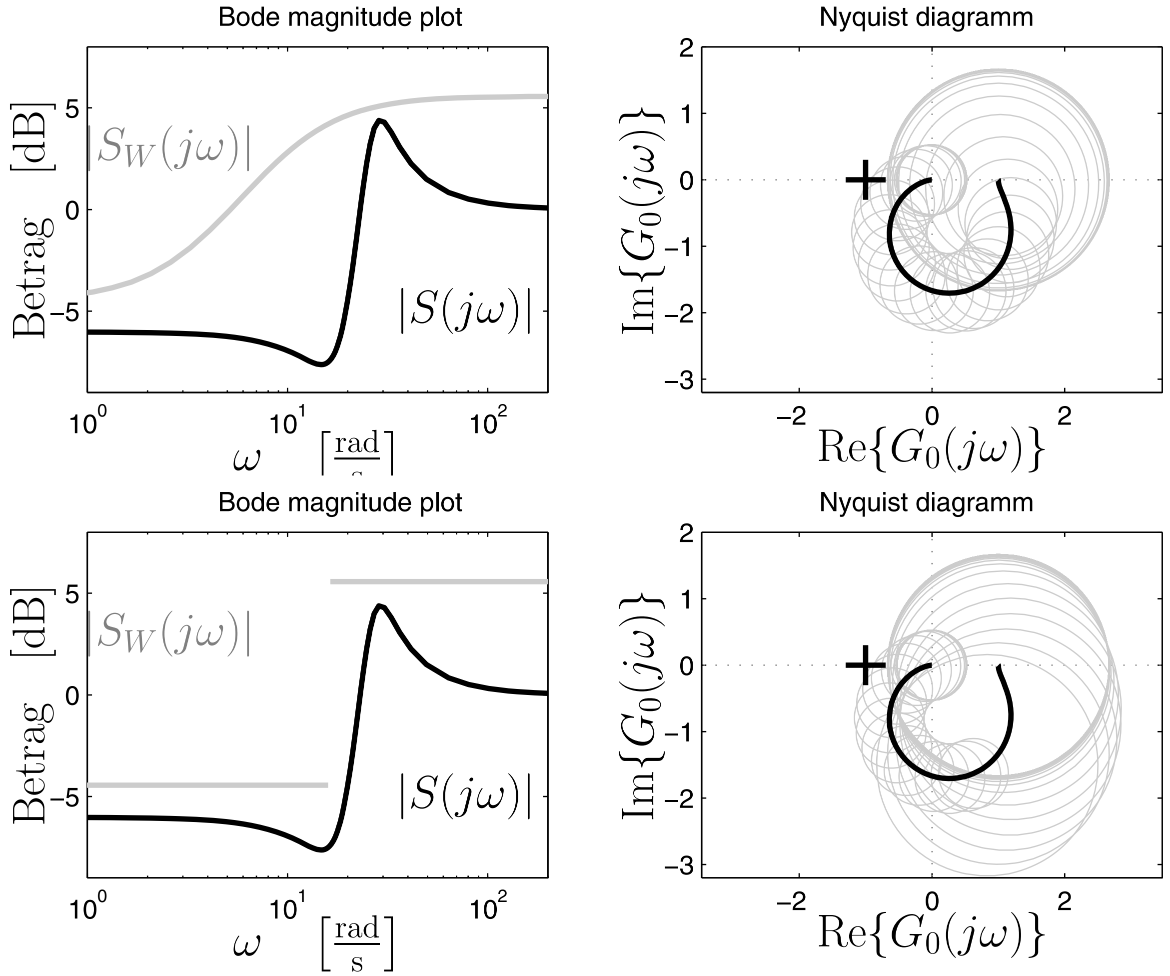

Figure 5.33. Graphic interpretation of the requirements on the sensitivity function in Bode Magnitude diagram.

$$|S_w(\mathrm{j}\omega)| = \begin{cases} l_g & \forall\ \ \omega \in [0, \omega_C] \\ h_g & \forall\ \ \omega \in [\omega_C, \infty] \end{cases} \tag{5.3.16}$$

$$\left.\begin{array}{rcl} |S(j\omega)| & < & |S_W(j\omega)| \\ |T(j\omega)| & < & |T_W(j\omega)| \\ |S_K(j\omega)| & < & |S_{K,W}(j\omega)| \\ |S_G(j\omega)| & < & |S_{G,W}(j\omega)| \end{array}\right\} \forall\, \omega \in [\omega^-, \omega^+]\,. \tag{5.3.35}$$

Robust $\boldsymbol{\mathcal{B}}$-stability: The family of frequency response magnitudes $\{|G(\mathrm{j}\omega, \boldsymbol{q})| | \boldsymbol{q} \in Q\}$ lies in a desired region $\boldsymbol{\mathcal{B}}$.

First $\boldsymbol{\mathcal{B}}$**-stability** for a fixed $\boldsymbol{q}$

$$|G(\mathrm{j}\omega)| < |F(\mathrm{j}\omega)| \text{ for all } \omega \in [\omega^{-}, \omega^{+}]$$

$$\left|\frac{G(\mathrm{j}\omega)}{F(\mathrm{j}\omega)}\right|^2 < 1 \leftrightarrow \xi < 0$$

$$\xi := \mathrm{Re}\left(\frac{G(\mathrm{j}\omega)}{F(\mathrm{j}\omega)}\right)^2 + \mathrm{Im}\left(\frac{G(\mathrm{j}\omega)}{F(\mathrm{j}\omega)}\right)^2 - 1 = \frac{p_1(\omega)}{den(\omega)} < 0$$

Robust $\mathcal{B}$-stability

$$\frac{p_1(\omega, \boldsymbol{q})}{d(\omega, \boldsymbol{q})} < 0 \quad \text{for all } \boldsymbol{q} \in Q$$

Point boundary condition at discontinuities and frequency endpoints of $|F(\mathrm{j}\omega)|$ for fixed $\omega = \omega^*$

$p_1(\omega^*, \boldsymbol{q}) = 0$

Tangent boundary condition at a frequency ω^*

$p_1(\omega^*, \boldsymbol{q}) = 0$

and

$$\frac{d}{d\omega} \; \frac{p_1}{den} \; = \; \frac{p_1' den - p_1 den'}{den^2}$$

equivalently

$$p_2(\omega^*, \boldsymbol{q}) = p_1'(\omega^*, \boldsymbol{q}) = 0 \quad \text{(because } p_1 = 0\text{)}$$

Map the condition

$$\begin{aligned} p_1(\omega, \boldsymbol{q}) &= 0 \\ p_2(\omega, \boldsymbol{q}) &= 0 \end{aligned}$$

for a grid on $\omega \in [\omega^-, \omega^+]$ to (q_1, q_2)-plane

Example

$$G(s, \omega_0, D) = \frac{\omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2}\,,\ \omega_0 > 0,\ D > 0$$

$$|G^{-}(\mathrm{j}\omega)| < |G(\mathrm{j}\omega, \omega_0, D)| < G^{+}(\mathrm{j}\omega)$$

$$\begin{aligned} G^{-}(s) &= \frac{90}{s^2 + 14s + 100} \\ G^{+}(s) &= \frac{1375}{2(s^2 + 5s + 625)} \end{aligned}$$

Point boundary conditions

$\omega = 0$ no point condition

$\omega = \infty$ $\omega_0 = 10\sqrt{9/10}$ and $\omega_0 = 25\sqrt{11/10}$

(dashed line)

Tangent boundary conditions

Two polynomials in ω_0^2 and D^2

(Solid lines)