A Warning Example

$$p(s,q_1,q_2) = (2+r^2+6q_1+6q_2+2q_1q_2)+(2+q_1+q_2)s + (2+q_1+q_2)s^2+s^3. \quad (9.1.1)$$

$q_1>0, q_2>0 \to a_0>0, a_1>0, a_2>0$

Stability condition

$a_1a_2-a_0=(q_1-1)^2+(q_2-1)^2-r^2>0$

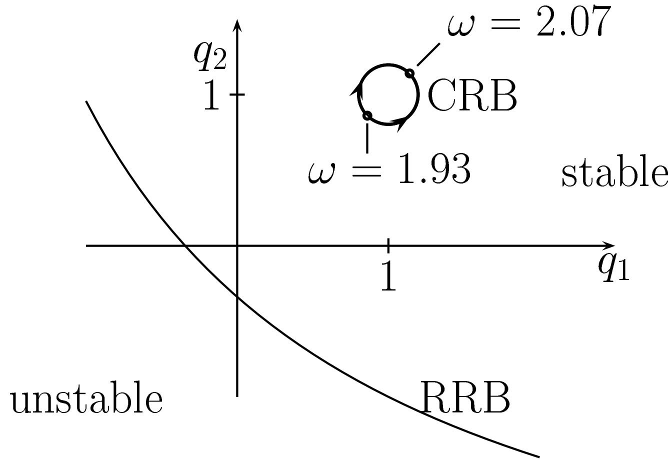

Fig. 4.5: The polynomial is unstable inside the circle

For $r \to 0$ isolated unstable point (IUP) at $q_1=1, q_2=1$. Choose arbitrary $Q-box$ that contains the IUP. The Q box cannot suffice for stability test!

For $r = 0.2$ we had

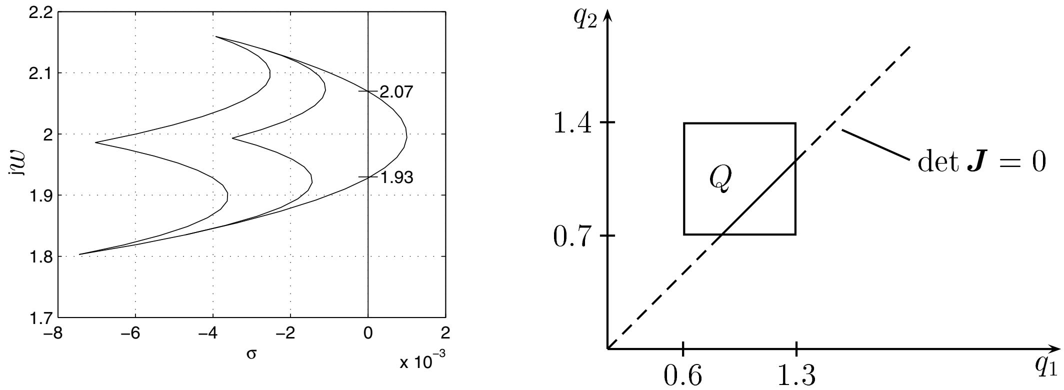

Fig. 4.7: The image of points on $\boldsymbol{J} = 0$ inside Q is the right hand boundary of the root set

Jacobian $\det J = 0$ for $q_1 = q_2$.

For $r = 0$ the root set touches the imaginary axis at $\omega = 2$.

Value set of $p(\mathrm{j}2, q_1, q_2), \boldsymbol{q} \in Q$-box

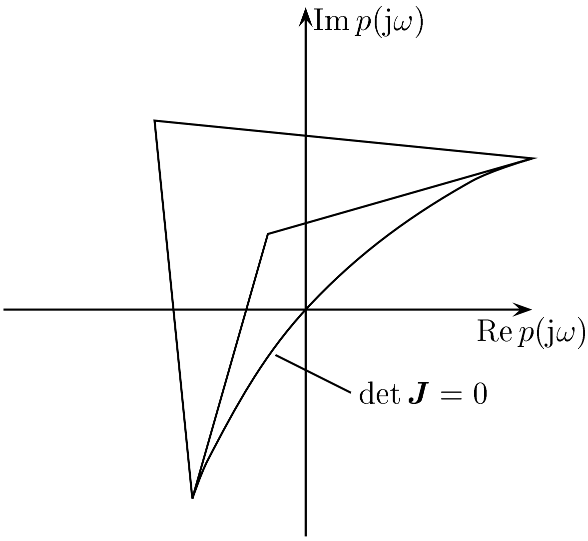

Fig. 9.1: Value set of the warning example at $\omega = 2$ (not to scale)

Zero exclusion at all other ω.

Chord approximation

Bilinear term $q_1 q_2$ is linear for fixed q_1 or fixed q_2.

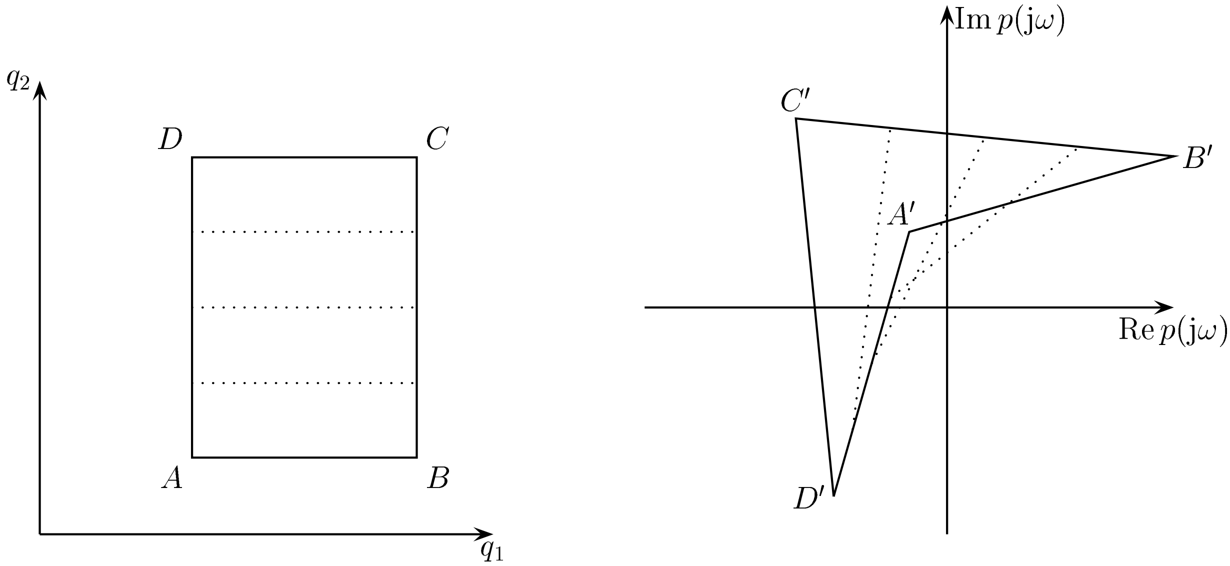

Fig. 9.2: Chord approximation of $\det \boldsymbol{J} = 0$ (not to scale).

Refine by further bisection

a) If the image $A'B'C'D'$ of the rectangle $ABCD$ is a convex quadrangle, then this is the value set.

b) If $A'B'C'D'$ is not convex, then the value set is contained in the convex hull, in Fig. 9.2 this is the triangle $B'C'D'$. This result will be generalized in Section 9.2.

More than two parameters

$$p(s, \boldsymbol{q}) = \prod_{i=1}^{3} (s + q_i) \tag{9.1.4}$$

$s = 0.5\mathrm{j}$, $q_i \in [-\sqrt{3}\,;\ \sqrt{3}]$, $i = 1, 2, 3$.

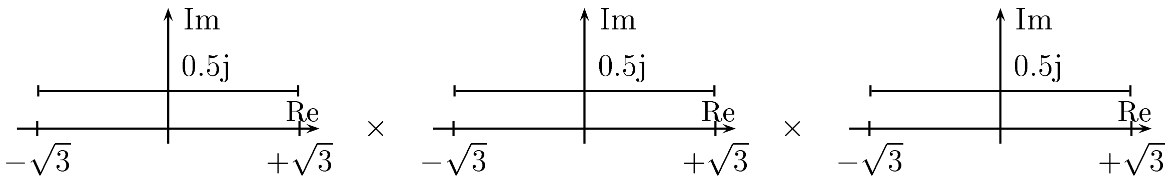

Fig. 9.3: Value sets of the subpolynomials

$$\begin{aligned} J(\omega, \boldsymbol{q}) &= \begin{bmatrix} \dfrac{\partial \mathrm{Re} p}{\partial q_1} & \dfrac{\partial \mathrm{Re} p}{\partial q_2} & \dfrac{\partial \mathrm{Re} p}{\partial q_3} \\ \dfrac{\partial \mathrm{Im} p}{\partial q_1} & \dfrac{\partial \mathrm{Im} p}{\partial q_2} & \dfrac{\partial \mathrm{Im} p}{\partial q_3} \end{bmatrix} \\ &= \begin{bmatrix} q_2 q_3 - \omega^2 & q_1 q_3 - \omega^2 & q_3 q_1 - \omega^2 \\ q_2 + q_3 & q_1 + q_3 & q_2 + q_1 \end{bmatrix} \end{aligned}$$

Extremum condition

$$\operatorname{rank} \boldsymbol{J}(\omega, \boldsymbol{q}) < 2 \tag{4.5.4}$$

satisfied for $q_1 = q_2 = q_3$.

This part of the boundary is generated by $(\sigma + 0.5\mathrm{j})^3$, $\sigma \in [-\sqrt{3}; \sqrt{3}]$

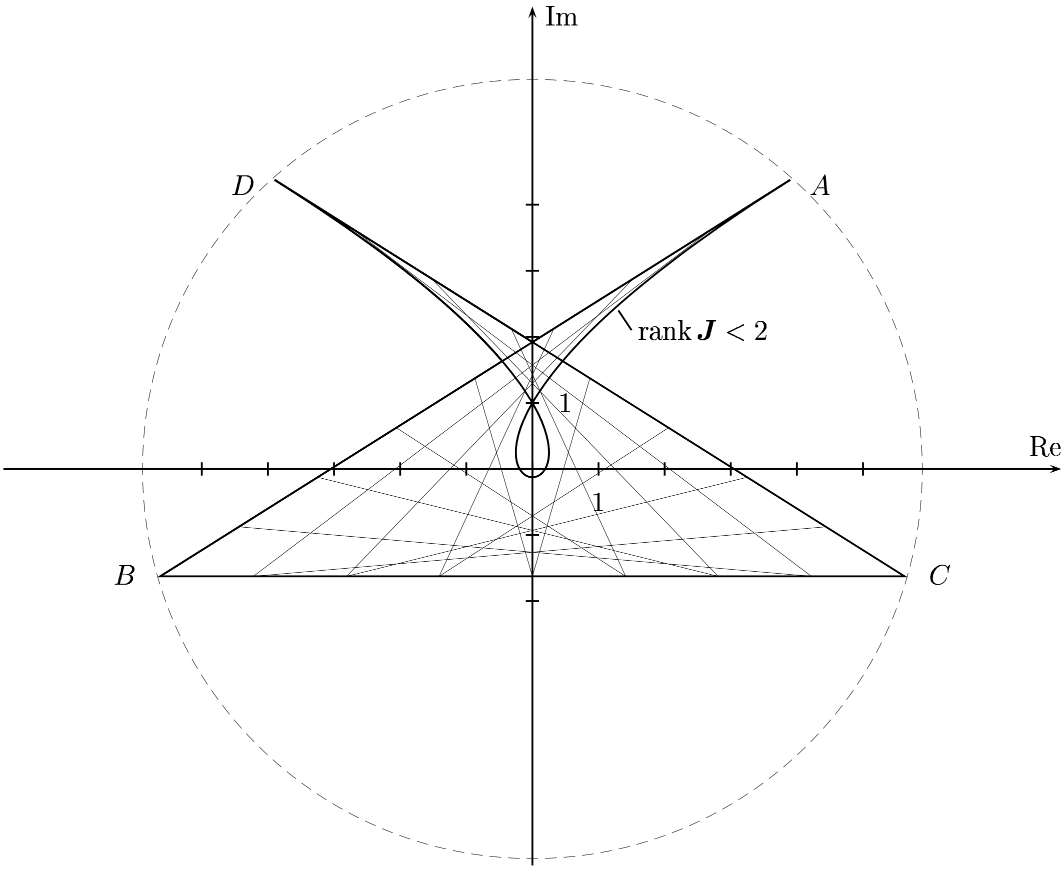

Fig. 9.4: Value set of the polynomial $\prod_{i=1}^{3}(s+q_i)$ for $s = 0.5\mathrm{j}$ and $q_i \in [-\sqrt{3}\,;\,\sqrt{3}]$

The convex hull of the value set is the rectangle ADBC.

Theorem 9.4 (Mapping theorem of Desoer)

The convex hull of the value set $\mathcal{P}(\mathrm{j}\omega^*, Q)$ of a polynomial with multilinear coefficient functions is the convex hull of the images of the vertices of Q.

$\square$

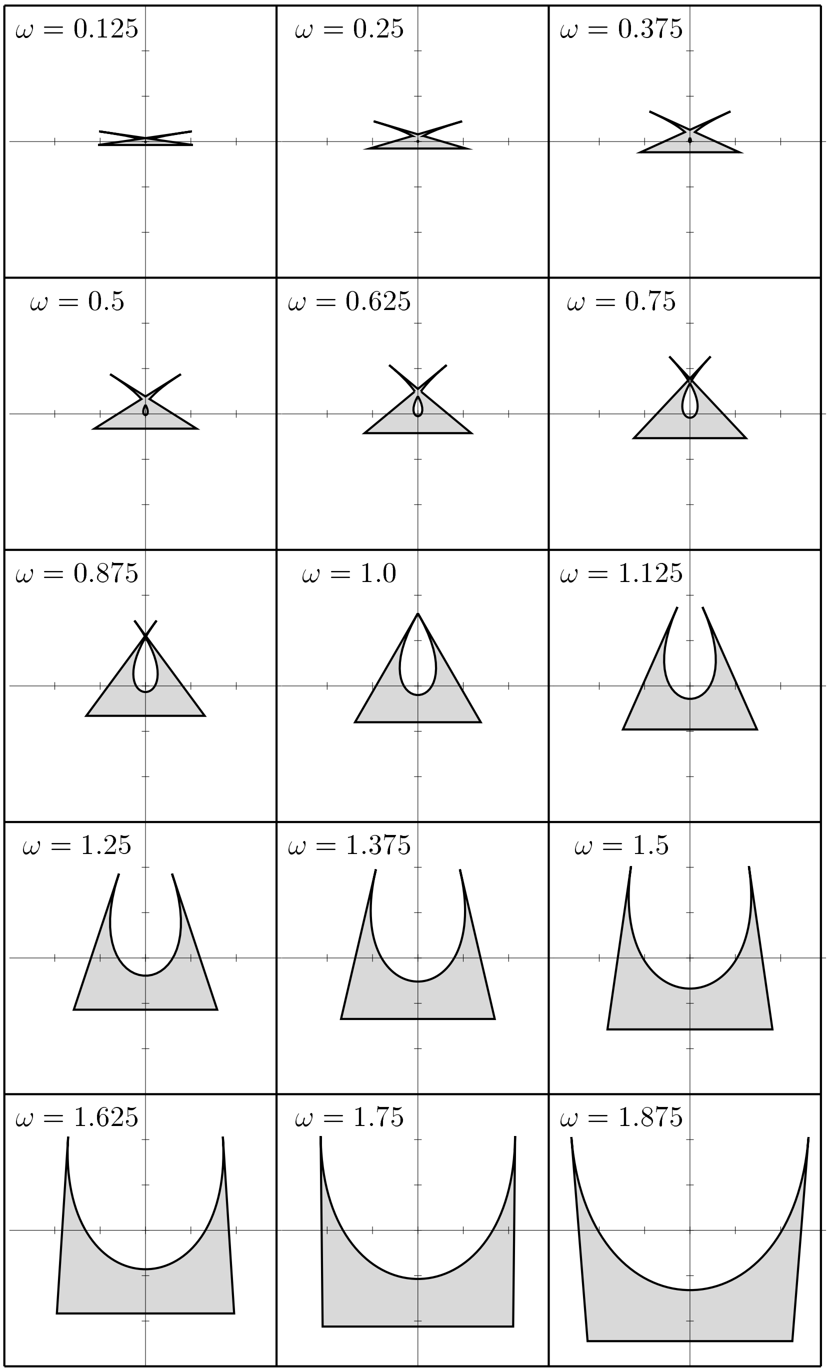

Fig. 6.8: Value sets of the uncertain polynomial (6.2.7)

Example 9.5

Crane with $\ell \in [8\,;\,16][\mathrm{m}]$, $m_C \in [100\,,\,2000][\mathrm{kg}]$,

$m_L = 2000$ [kg]

Control law

$u = -[500\ \ 100\ \ -100\ \ 0]\boldsymbol{x}$

$p(s, m_C, \ell) = 5000 + 1000s + (20\,100 + 500\ell + 10m_C)s^2 + 100\ell s^3 + \ell m_C s^4.$

Stable test point $\ell = 8[\mathrm{m}]$, $m_C = 1000[\mathrm{kg}]$.

Construct convex hull of value sets for an ω-grid.

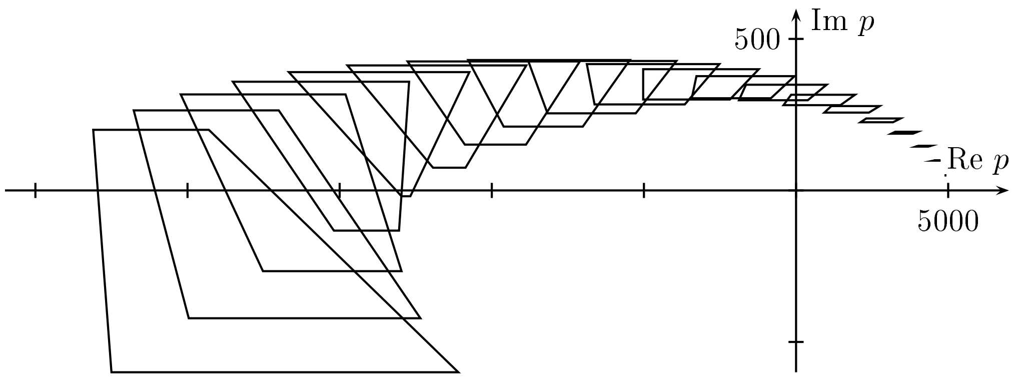

Fig. 9.7: Convex hulls for the given example with $\omega^* = k \cdot 0.05$, $k = 0, 1, \ldots, 20$

Origin excluded from convex hull and therefore from value set. The system is robustly stable.

Convex hull condition does not hold for polynomial coefficient functions!

$p(s, q_1) = s + q_1^2 - 6q_1 + 8, \quad q_1 \in [1\,;\,5]$

Endpoints stable:

$p(s, 1) = s + 3$

$p(s, 5) = s + 3$

For each $\omega = \omega^\star$ the convex hull is just the point $\mathrm{j}\omega^\star + 3$.

But interior point

$p(s, 3) = s - 1$

is unstable.

Tree-structured Value Set Construction

Crane example

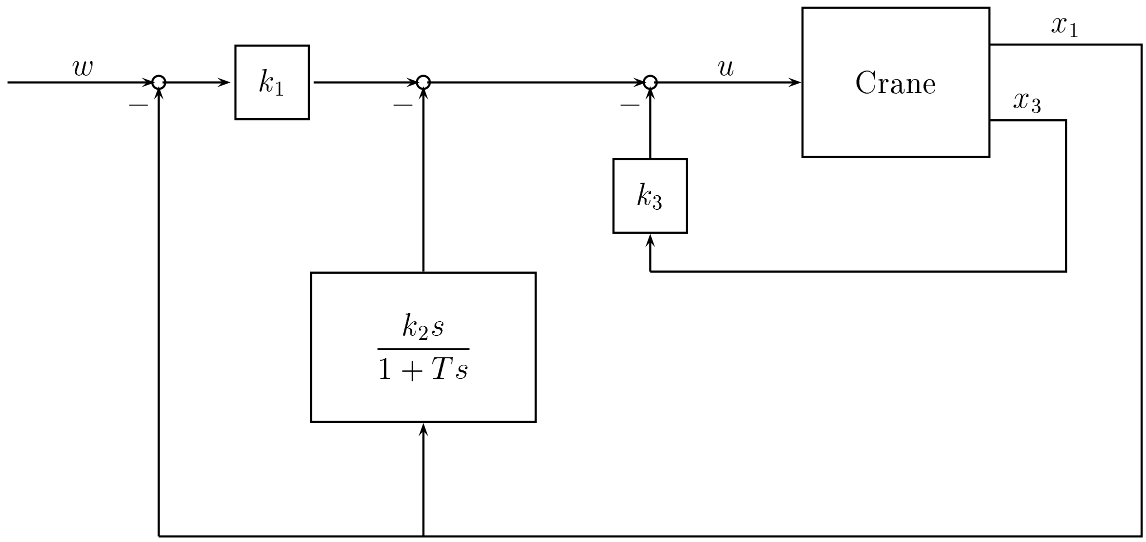

Fig. 1.8: Robust controller structure for the crane with $k_1 > 0$, $k_2 > 0$, $k_3 < 0$.

Characteristic polynomial for fixed T

$$p(s, \ell, m_L, m_C, k_1, k_2, k_3) = (s^2\ell + g)[(s^2 m_C + k_1)(1 + Ts) + k_2 s] + s^2(m_L g - k_3)(1 + Ts). \quad (9.3.1)$$

Each of the six parameters appears only once.

Therefore the value sets of $(s^2\ell+g)$, $(s^2 m_C+k_1)$, $(m_L g-k_3)$ and k_2 be constructed **independently** and combined to the value set $p(s, \ell, m_L, m_C, k_1, k_2, k_3)$, where s is a fixed complex number on the boundary $\partial\Gamma$. For Hurwitz stability $s = \mathrm{j}\omega$.

$$p(s,\ell,m_L,m_C,k_1,k_2,k_3) \;=\; (s^2\ell+g)[(s^2m_C+k_1)(1+Ts)+k_2s] + s^2(m_Lg-k_3)(1+Ts). \qquad (9.3.1)$$

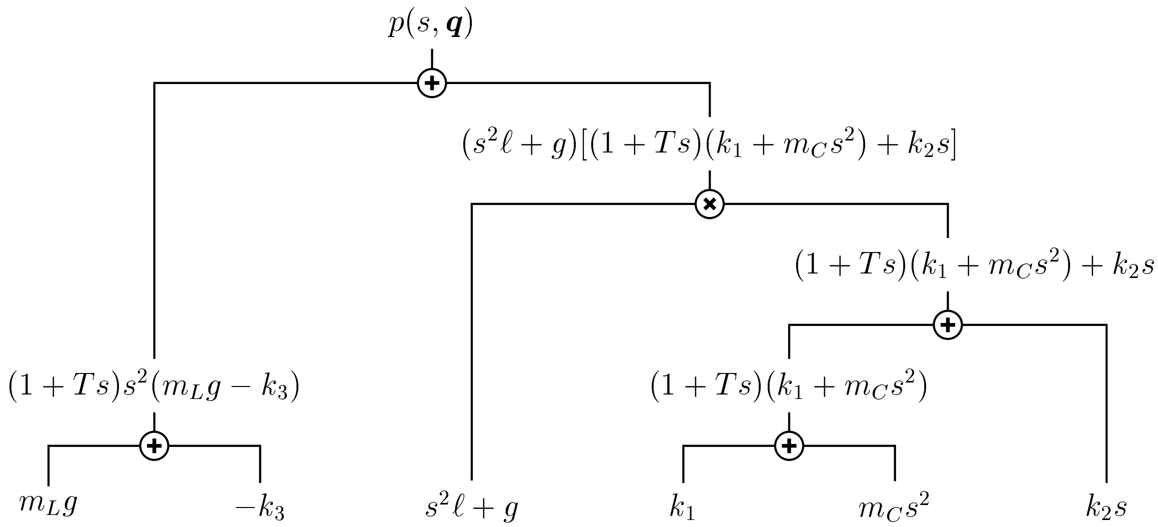

Fig. 9.8: Tree-structure of the characteristic polynomial (9.3.1)

Let $s = \mathrm{j}\omega$

$k_1 + m_C s^2 \;\rightarrow\; k_1 - m_C\omega^2$

real interval $[k_1^- - m_C^+\omega^2\,;\; k_1^+ - m_C^-\omega^2]$

Rotate and scale by multiplication by the complex number $(1 + \mathrm{j}\omega T)$.

$\rightarrow$ line segment $p_1 p_2$

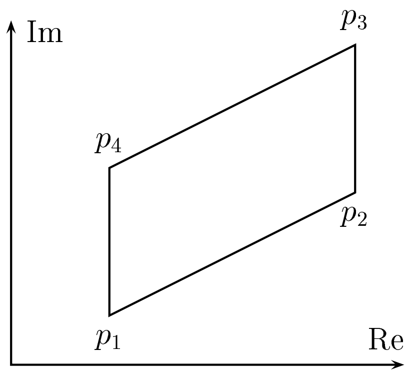

Fig. 9.9: Value set of $(1 + \mathrm{j}\omega T)(k_1 + m_C s^2) + k_2 s$ for a fixed frequency $s = \mathrm{j}\omega$

Add $k_2\mathrm{j}\omega$

$\rightarrow$ parallelogram $p_1 p_2 p_3 p_4$.

Similarly construct value sets of $(s^2\ell + g)$ and $(1 + \mathrm{j}\omega T)(m_L g - k_3)$, the combine to

$$\begin{aligned} p(s, \ell, m_L, m_C, k_1, k_2, k_3) \;&=\; (s^2\ell + g)[(s^2 m_C + k_1)(1 + Ts) + k_2 s] \\ &\quad + s^2(m_L g - k_3)(1 + Ts). \qquad (9.3.1) \end{aligned}$$

Need tools for complex set addition and multiplication.

Mass-spring-damper systems

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{x}} + \boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{x}} + \boldsymbol{K}(\boldsymbol{q})\boldsymbol{x} = \boldsymbol{u} \tag{9.3.2}$$

Usual way to arrive at a state space model: Premultiply by inverse mass matrix $\boldsymbol{M}(\boldsymbol{q})$. This “smears” the uncertain masses all over the model. Better Laplace transform

$$p(s, \boldsymbol{q}) = \det\left[\boldsymbol{M}(\boldsymbol{q})s^2 + \boldsymbol{D}(\boldsymbol{q})s + \boldsymbol{K}(\boldsymbol{q})\right] \tag{9.3.3}$$

Its value set for $s = \mathrm{j}\omega$ is

$$p(\mathrm{j}\omega, \boldsymbol{q}) = \det\left[\boldsymbol{K}(\boldsymbol{q}) - \omega^2\,\boldsymbol{M}(\boldsymbol{q}) + \mathrm{j}\omega\boldsymbol{D}(\boldsymbol{q})\right]. \tag{9.3.4}$$

Example 9.9

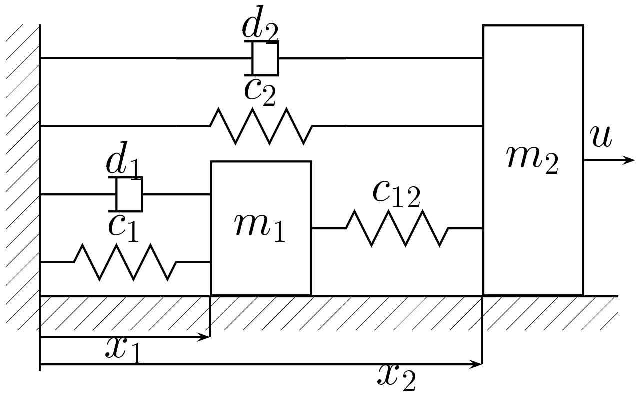

Fig. 9.10: Schematic representation of a mechanical system

$$
\begin{aligned}
m_1\ddot{x}_1 + d_1\dot{x}_1 + c_1x_1 + c_{12}(x_1 - x_2) &= 0, \\
m_2\ddot{x}_2 + d_2\dot{x}_2 + c_2x_2 + c_{12}(x_2 - x_1) &= u,
\end{aligned}
$$

and their Laplace transform is

$$
\begin{bmatrix} m_1s^2 + d_1s + c_1 + c_{12} & -c_{12} \\ -c_{12} & m_2s^2 + d_2s + c_2 + c_{12} \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ u(s) \end{bmatrix}
$$

$$
p(s, \boldsymbol{q}) = p_1(s, m_1, d_1, c_1, c_{12}) \cdot p_2(s, m_2, d_2, c_2, c_{12}) - c_{12}^2, \tag{9.3.5}
$$

$$
p_i(s, m_i, d_i, c_i, c_{12}) = m_is^2 + d_is + c_i + c_{12}, \quad i = 1, 2. \tag{9.3.6}
$$

Tree structure only for $c_{12} = \text{const.} \rightarrow$ Grid c_{12}.

Consider a fixed c_{12}.

The value set of

$p_i(\mathrm{j}\omega, m_i, d_i, c_i) = -m_i\omega^2 + d_i\mathrm{j}\omega + c_i + c_{12}$

is a rectangle

Multiply two rectangles

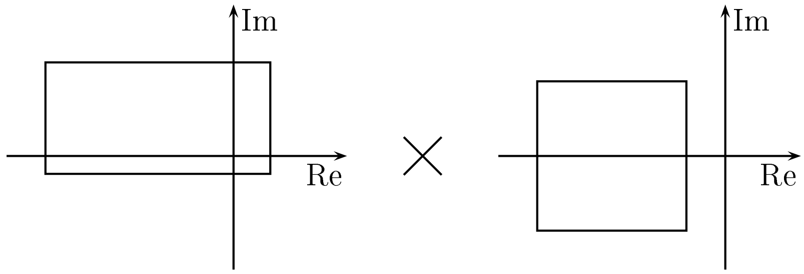

Fig. 9.11: Computation of the value set by multiplication of two rectangles in the complex plane

and subtract c_{12}^2.

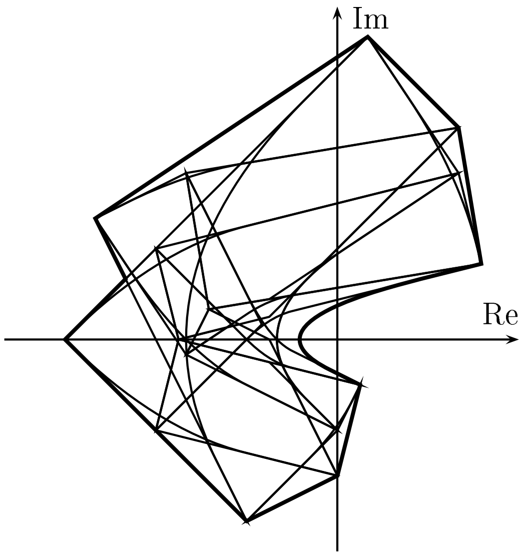

Fig. 9.12: Value set for the mechanical system of Fig. 9.10 at $\omega^* = 1$

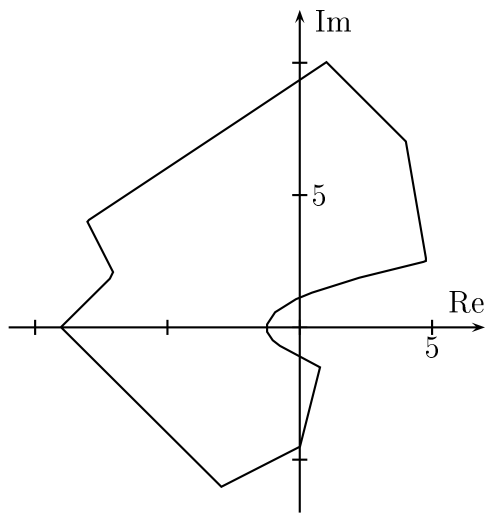

Fig. 9.13: Contour of the value set of Fig. 9.12

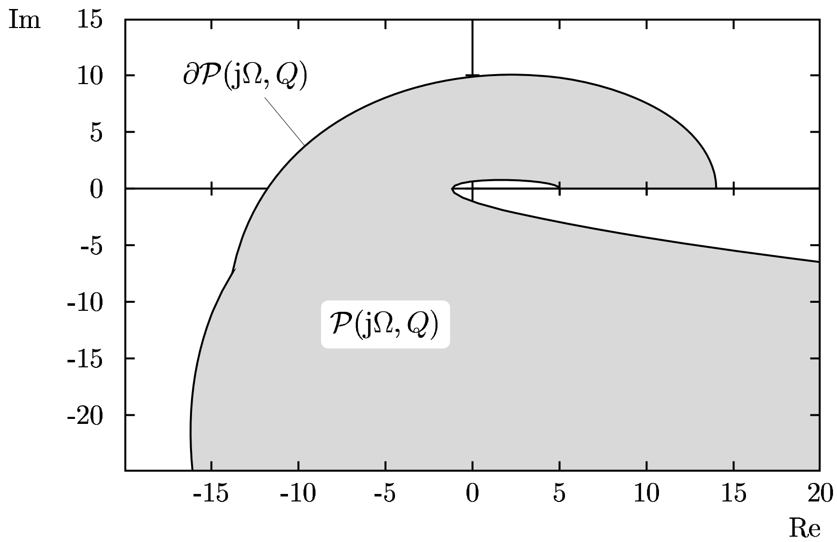

Fig. 9.14: Union of the value sets of the mechanical system

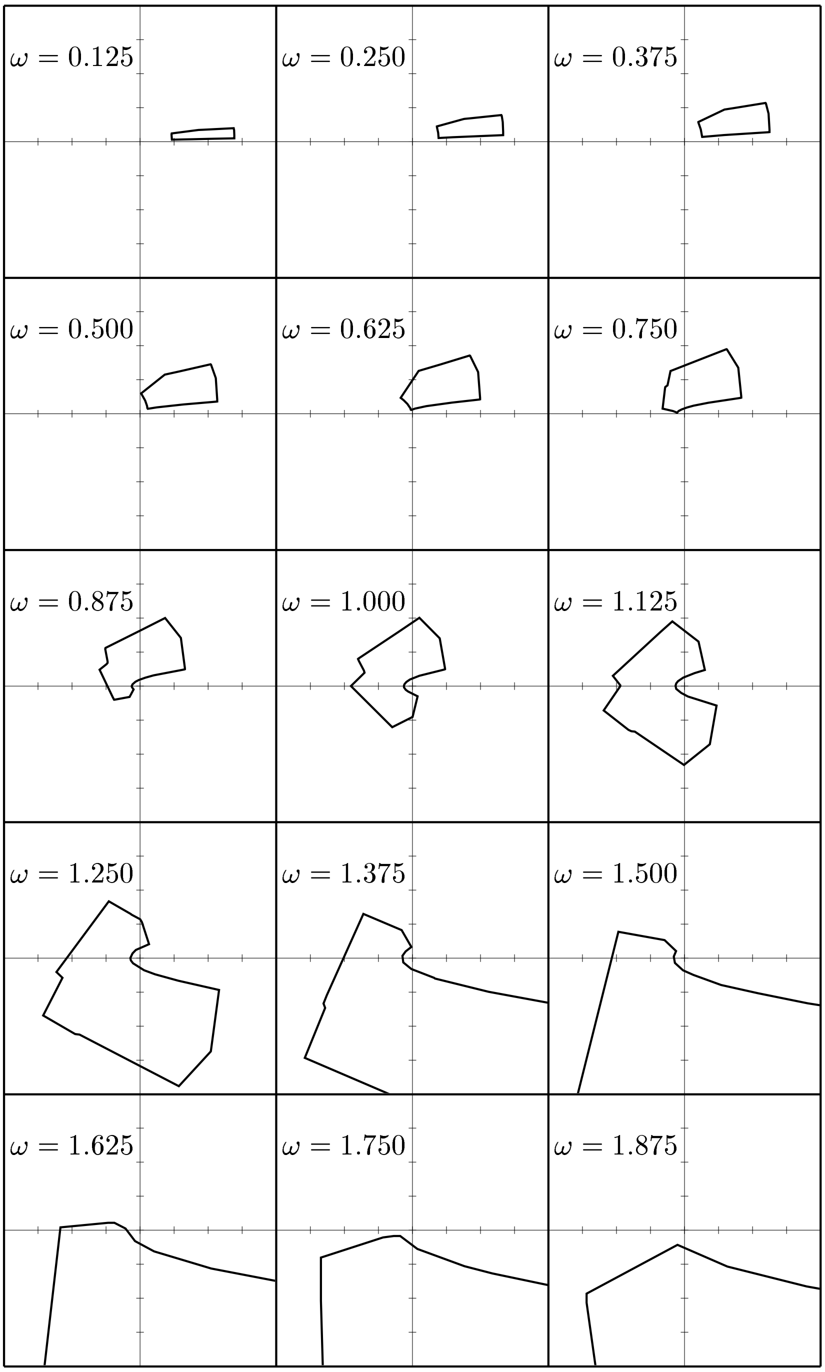

Fig. 9.15: Scenes from a value set animation

Value set addition

$$\mathcal{C} = \mathcal{A} + \mathcal{B} = \{a + b \,|\; a \in \mathcal{A},\; b \in \mathcal{B}\}, \tag{9.4.1}$$

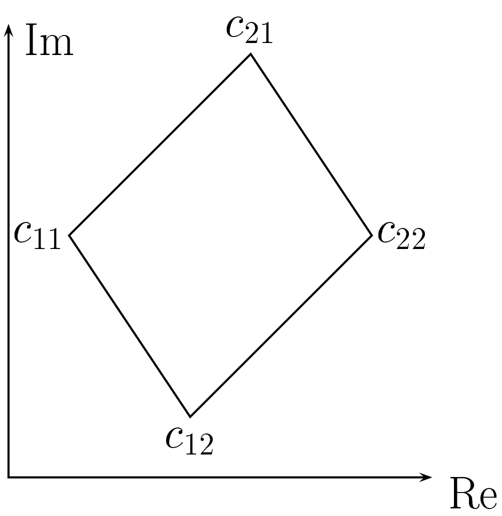

Fig. 9.17: Sum of two line segments

Here $\partial\mathcal{C} = \partial\mathcal{A} + \partial\mathcal{B}$.

In general:

$\partial\mathcal{C} \subset \partial\mathcal{A} + \partial\mathcal{B}$

Value set multiplication

$$\mathcal{C} = \mathcal{A} \cdot \mathcal{B} = \{a \cdot b \,|\, a \in \mathcal{A},\ b \in \mathcal{B}\}. \qquad (9.4.2)$$

Multiply two line segments

$$\begin{aligned} a &= a_1 + \alpha(a_2 - a_1),\ \alpha \in [0\,;\ 1] \text{ and} \\ b &= b_1 = \beta(b_2 - b_1),\ \beta \in [0\,;\ 1]. \end{aligned}$$

$c = a \cdot b$ is an uncertain bilinear term in α and β.

$$J(\alpha, \beta) = \begin{vmatrix} \dfrac{\partial \mathrm{Re}\, c(\alpha, \beta)}{\partial \alpha} & \dfrac{\partial \mathrm{Re}\, c(\alpha, \beta)}{\partial \beta} \\ \dfrac{\partial \mathrm{Im}\, c(\alpha, \beta)}{\partial \alpha} & \dfrac{\partial \mathrm{Im}\, c(\alpha, \beta)}{\partial \beta} \end{vmatrix} \qquad (9.4.7)$$

Boundaries of set $\mathcal{C} = \mathcal{A} \cdot \mathcal{B}$ result from

1) $\alpha \in [0\,;\ 1]$ for $\beta = 0$ and 1

2) $\beta \in [0\,;\ 1]$ for $\alpha = 0$ and 1

3) The parabola $\det J(\alpha, \beta) = 0$

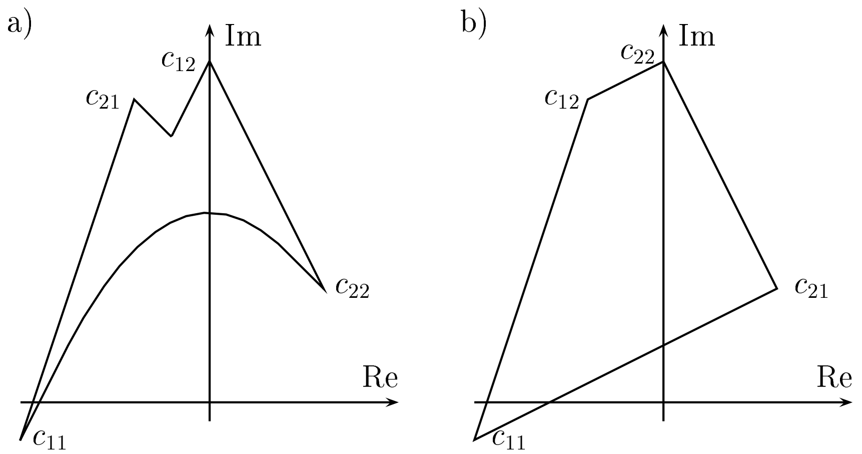

Fig. 9.18: Products of two line segments: a) the Jacobian determinant vanishes for $\alpha \in [0;\ 1]$ and $\beta \in [0;\ 1]$, b) the Jacobian determinant does not vanish, i.e. the set is convex

Chord approximation

The parabola may be approximated by a chord of straight line segments.

Thereby, the boundary $\partial\mathcal{C}$ is described by straight line segments and prepared for the next operation.

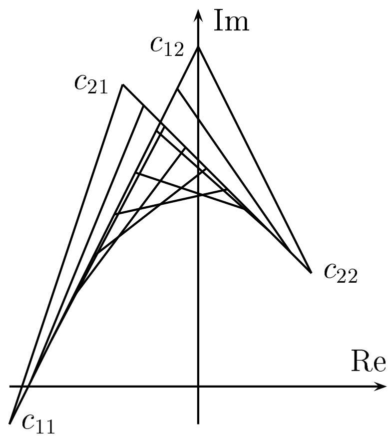

Fig. 9.19: Construction of the product of two line segments by approximation

Tree-structured decomposition (TSD)

Assume that the tree structure does not result from modelling (as in MSD systems).

Find a TSD (if it exists) of a parametric polynomial.

Define a partition of the uncertainties $q_1, q_2, \dots, q_\ell$ in $\boldsymbol{q}$. $\boldsymbol{q}$ is decomposed into subvectors $\boldsymbol{q}^{(1)}$ and $\boldsymbol{q}^{(2)}$ with no common elements, i.e. each uncertainty q_i belongs either to $\boldsymbol{q}^{(1)}$ or to $\boldsymbol{q}^{(2)}$.

A polynomial $p(s, \boldsymbol{q})$ is **sum decomposable**, if there exists a partition $\boldsymbol{q}^{(1)}$, $\boldsymbol{q}^{(2)}$ such that

$$p(s, \boldsymbol{q}) = p_1(s, \boldsymbol{q}^{(1)}) + p_2(s, \boldsymbol{q}^{(2)}) \tag{9.4.8}$$

A polynomial $p(s, \boldsymbol{q})$ is **product decomposable**, if there exists a partition $\boldsymbol{q}^{(1)}$, $\boldsymbol{q}^{(2)}$ such that

$$p(s, \boldsymbol{q}) = p_1(s, \boldsymbol{q}^{(1)}) \cdot p_2(s, \boldsymbol{q}^{(2)}) + p_0(s) \tag{9.4.9}$$

The "subpolynomials" p_1 and p_2 may be further decomposable.

Example 9.12

$$p(s, \boldsymbol{q}) = (q_1 q_2 + q_3 + q_3^2) + (q_1 - 4q_2 + q_3 + 3)s \qquad (9.4.10)$$

1. Sum decomposition

$$p(s, \boldsymbol{q}) = p_1(s, q_1, q_2) + p_2(s, q_3)$$

with

$$\begin{aligned} p_1(s, q_1, q_2) &= q_1 q_2 + (q_1 - 4q_2)s, \\ p_2(s, q_3) &= q_3 + q_3^2 + q_3 s + 3s. \end{aligned}$$

2. Product decomposition

$$p_1(s, q_1, q_2) = p_{11}(s, q_1) \cdot p_{12}(s, q_2) + p_0(s),$$

add $-4s^2$ to p_1 and include $+4s^2$ in $p_0(s)$

$$\begin{aligned} p_{11}(s, q_1) &= q_1 - 4s, \\ p_{12}(s, q_2) &= q_2 + s, \\ p_0(s) &= 4s^2. \end{aligned}$$

The degree of the subpolynomials in s may be higher than the degree of $p(s, \boldsymbol{q})$.

For s just a complex number on $\partial\Gamma$ is substituted.

Sum decomposition is easy by collecting q_i terms.

Product decomposition test for q_i in $\boldsymbol{q}^{(1)}$

$$\begin{aligned} p(s, \boldsymbol{q}) &= p_1(s, \boldsymbol{q}^{(1)}) \cdot p_2(s, \boldsymbol{q}^{(2)}) + p_0(s) \\ \frac{\partial p(s, \boldsymbol{q})}{\partial q_i} &= \frac{\partial p_1(s, \boldsymbol{q}^{(1)})}{\partial q_i} \cdot p_2(s, \boldsymbol{q}^{(2)}) \end{aligned}$$

Factorize symbolically.

Nyquist value sets

$$G(s, \boldsymbol{n}, \boldsymbol{d}) = \frac{n(s, \boldsymbol{n})}{d(s, \boldsymbol{d})} \tag{9.5.1}$$

no common uncertainties in $\boldsymbol{n}$ and $\boldsymbol{d}$.

Invert the value set of $d(s, \boldsymbol{d})$ and multiply by value set of $n(s, \boldsymbol{n})$

$$\mathcal{A}^{-1} = \{1/a \mid a \in \mathcal{A}\}, \qquad (0 \notin \mathcal{A}) \tag{9.5.2}$$

Invert only boundary

$$\partial(\mathcal{A}^{-1}) = (\partial\mathcal{A})^{-1} \tag{9.5.3}$$

A straight line segment (e.g. chord element approximating $\partial(\mathcal{A}^{-1})$)

$z = x + \mathrm{j}y, \;\; 2ax + 2by = 1$

is mapped to a circular arc with center $(a; -b)$ and radius $\sqrt{a^2 + b^2}$. The complete circle passes through the origin

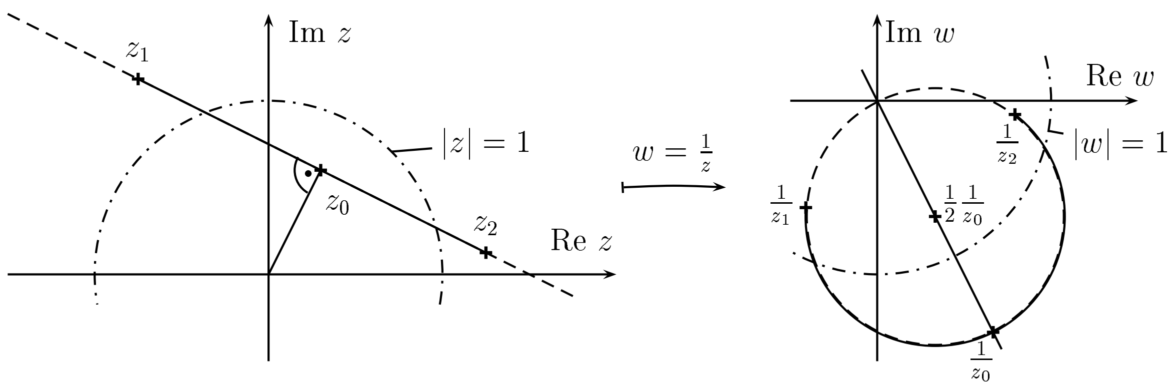

Fig. 9.20: Inversion of a line segment

MSD system with feedback controller

6 parameters m_i, d_i, c_i, $i = 1, 2$

Open-loop Nyquist set

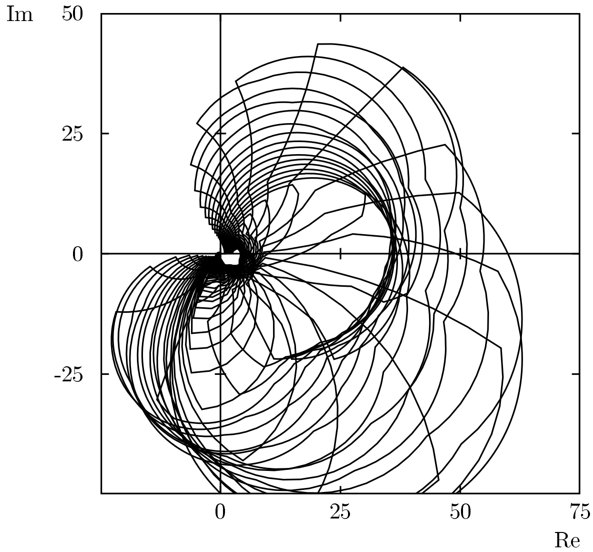

Fig. 9.22: Union of Nyquist sets for a grid on ω

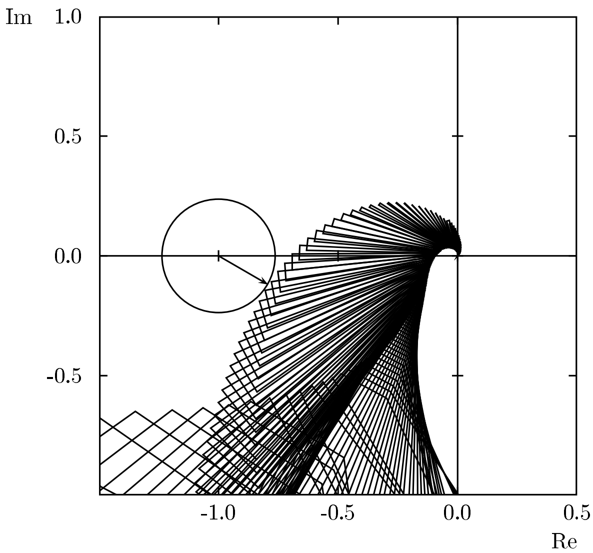

Fig. 9.23: Detailed view of the union of Nyquist sets

The Stability Profile

= right hand boundary of the root set.

The polynomial family

$$P(s, Q) = \{p(s, \boldsymbol{q}) \,|\, \boldsymbol{q} \in Q\}$$

is σ-stable (all $\mathrm{Re} s_i < \sigma_0$), if and only if

1. There is a σ_0-stable polynomial in the family.

2. The value set $P(\sigma_0 + \mathrm{j}\omega, Q)$ excludes the origin at all frequencies $\omega \geq 0$.

The substitution of $s = \sigma_0 + \mathrm{j}\omega$ instead of $s = \mathrm{j}\omega$ does not destroy a TSD.

MSD-system for $\omega^\star = 1$.

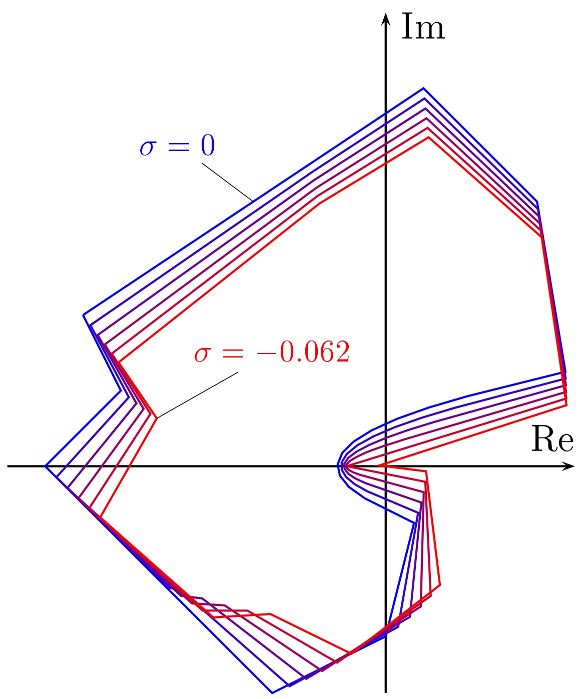

Fig. 9.27: The real part is shifted to the left until zero is no longer excluded for $\sigma_0 = -0.062$

Repeat for a grid on ω

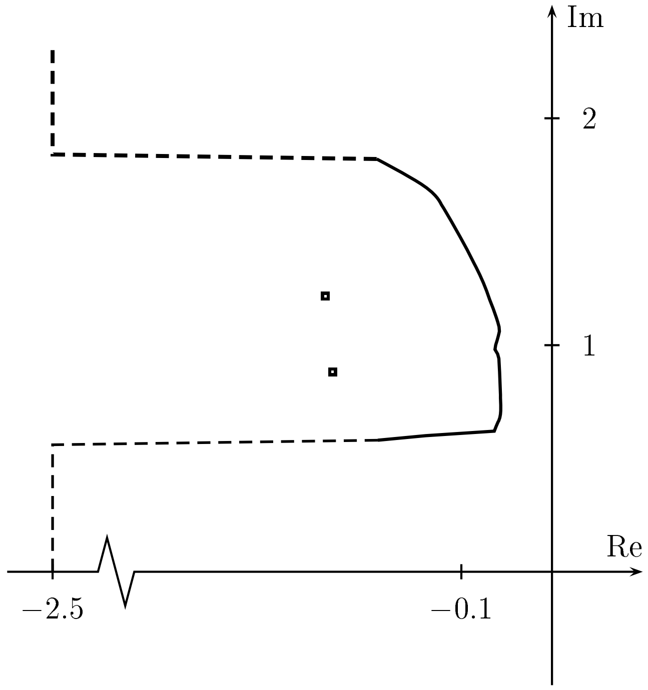

Fig. 9.28: Stability profile for the mass-spring-damper system of Fig. 9.10

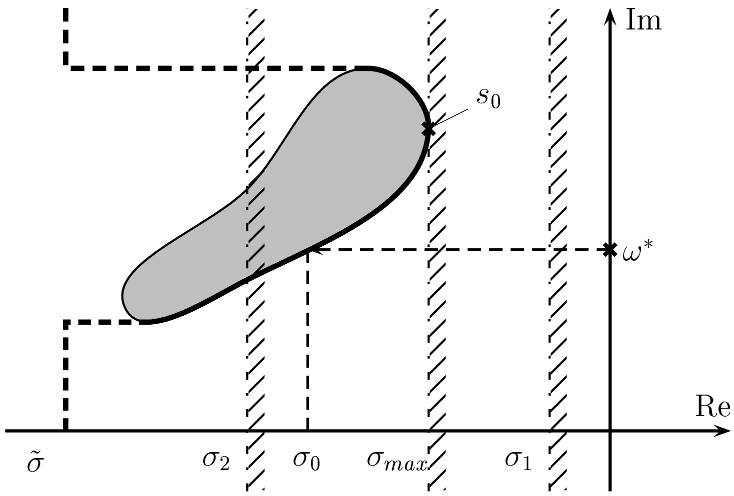

Fig. 9.29: The real value σ_0 is founded by repeated bisection between a Γ-stable value σ_1 and a Γ-unstable value σ_2. Then $\sigma_{\max} = \max_{\omega} \sigma_0(\omega)$