Cost Functional Minimizing Sliding Mode Control Design

Christian Fischer, Heide Brandtstädter, Vadim Utkin and Martin Buss

***Abstract*— In this paper, a three-step method is proposed to design a sliding mode control strategy, that provides optimal system dynamics with respect to a given cost functional. The method is based on methods of constrained continuous dynamic optimization. It allows a systematic control unit design and therefore reduces the time to find an appropriate sliding mode control law. The proposed technique is successfully applied to the position control of a three-phase permanent magnet synchronous motor. During operation, the dissipative power loss of the machine is minimized.**

I. INTRODUCTION

Systems in sliding mode are very robust against disturbances. In addition to that, controllers that enforce systems to sliding mode are discontinuous ones and can often be realized simpler or better than continuous controllers.

However, using finite switching frequencies sliding mode control may result in chattering. This side-effect might be audible as a chattering sound but it may also lead to a vibration state, that might cause material fatigue in the system. Meanwhile, new switching devices providing high enough switching frequencies as well as some control engineering based countermeasures against this chattering problem have been developed.

Research of differential equations with discontinuous right hand sides and the occurrence of sliding modes in such systems have been carried out since the 1960's, mainly in the Soviet Union, but the principle has already been used in the 1930's [1], [2]. During the middle of 1980's, sliding mode control attracted more attention because of the rise of power semiconductors that began to replace relay switches and allowed higher switching frequencies.

The freedom in designing a sliding mode controller provides some margin for optimization. Several methods for sliding mode control design, which focus on selected optimization problems, have been developed: In [4] and [5] control strategies for linear time invariant systems are presented, in [6] a cost functional minimizing switching controller for a space craft is developed. Furthermore, in [7] an optimal fuzzy logic sliding mode controller is designed, which solves the chattering problem due to the smoothening property of the fuzzy set.

This article presents a method for sliding mode control design for nonlinear systems. In sliding mode the system dynamics follow an optimal trajectory which is calculated based on methods of constrained continuous dynamic optimization.

The remaining part of this article is organized as follows: Section 2 shortly presents the mathematical background of sliding modes in dynamic systems. Section 3 explains the design concept for a sliding mode controller that minimizes a given cost functional. This concept is applied to a real system in section 4. Section 5 concludes this article.

II. SLIDING MODE CONTROL

An arbitrary dynamic system with n-dimensional state

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ represents the system state and $\boldsymbol{z}$ summarizes the unknown internal and external disturbances, is considered. The system control input $\boldsymbol{u} \in \mathbb{R}^m$ is chosen as a discontinuous function of the state: $\boldsymbol{u}(\boldsymbol{x})$.

Sliding mode in a dynamic system is characterized by the equation

$$\boldsymbol{s} = \boldsymbol{0} \tag{2}$$

with $\boldsymbol{s}(\boldsymbol{x})$ being the switching function. Equation (2) determines the manifold where the state in sliding mode is forced to be. Once the state reaches the switching manifold, it will remain there. Therefore, for deriving the sliding mode equations

$$\dot{\boldsymbol{s}} = \boldsymbol{0} \tag{3}$$

is assumed as well. Condition (3) leads to the so called *equivalent control*. It proposes to substitute a continuous function $\boldsymbol{u_{eq}}$ for the discontinuous control $\boldsymbol{u}$ in the original system (1).

In order to enforce a system to sliding mode, the switching manifold $\boldsymbol{s} = \boldsymbol{0}$ has to be reached in finite time. Using a Lyapunov function

$$V \in \mathbb{R}^+ : \begin{cases} V = 0 & \text{for} \quad \boldsymbol{s} = \boldsymbol{0} \\ V > 0 & \text{for} \quad \boldsymbol{s} \neq \boldsymbol{0} \end{cases} \tag{4}$$

the sufficient condition satisfying the equation below

$$\dot{V} \leq -\epsilon\sqrt{V}, \epsilon > 0 \tag{5}$$

can be derived. This is called the *reaching condition*. If the switching controller $\boldsymbol{u}(\boldsymbol{x})$ fulfills condition (5), it is called a sliding mode controller [2].

III. COST FUNCTIONAL MINIMIZING SLIDING MODE CONTROL DESIGN

Development of a sliding mode controller is realized in two steps. First of all the switching function is selected, secondly the controller is designed. The switching function $\boldsymbol{s}(\boldsymbol{x})$ is arbitrary and the sliding mode controller $\boldsymbol{u}(\boldsymbol{x})$ has

Chr. Fischer, H. Brandtstädter, and M. Buss are with the Institute of Automatic Control Engineering (LSR), Technische Universität München, 80290 Munich, Germany. `chfm@gmx.de`, `heide.brandtstaedter@tum.de`, `m.buss@ieee.org`

V. Utkin is with the Department of Mechanical Engineering, The Ohio State University, Columbus, Ohio, USA `utkin@ece.osu.edu`

to fulfill the reaching condition (5). Thus both, $\boldsymbol{s}$ and $\boldsymbol{u}$, have to be selected. In some cases like linear systems of low dimension this may be rather straightforward, but in general, several design loops are necessary due to the complexity of the nonlinear control law. A more theoretical approach would be desirable.

The freedom in design of $\boldsymbol{s}$ and $\boldsymbol{u}$ makes both variables subject of optimization. Usually there exist a lot of different finite continuous control trajectories $\boldsymbol{u}(t),\, t \in [0, T]$ that lead the system trajectory from an initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$ to a target state $\boldsymbol{x}(T) = \boldsymbol{x}_T$. Among all these trajectories the control trajectory $\boldsymbol{u}^*(t)$ (and also the corresponding system trajectory $\boldsymbol{x}^*(t)$) is chosen that minimizes an arbitrary cost functional $J(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}, T)$. The control trajectory $\boldsymbol{u}^*(t)$ is optimal for the modeled system. But even if the model is just an approximation of the real system, the solution found using this method would then be an appropriate initial guess.

In this paper a design method for sliding mode controllers, which follow an optimal trajectory $\boldsymbol{x}^*(t)$ in sliding mode, is developed. This leads to optimization of the system motion with respect to a given cost functional. In first step, an optimal continuous feedback control law $\boldsymbol{u}^*(\boldsymbol{x})$ for the system (1) with respect to a given cost functional is found. Since the disturbance $\boldsymbol{z}$ is unknown, this has to be considered for the *nominal system*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{z}}, \boldsymbol{u}) \tag{6}$$

where $\boldsymbol{z}$ in (1) is replaced by a nominal value $\bar{\boldsymbol{z}}$. Where statistical value of disturbance is known, $\bar{\boldsymbol{z}}$ is the expected or mean value of $\boldsymbol{z}$ and may be time varying. In second step, the switching function $\boldsymbol{s}^*(\boldsymbol{x})$ is calculated so that it forces the system to move along the optimal trajectory $\boldsymbol{x}^*$. The third step consists of finding an appropriate switching controller $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{s}^*)$ which fulfills the reaching condition (5) and enforces sliding mode in the manifold $\boldsymbol{s}^* = \boldsymbol{0}$.

Before the three-step optimization process is described in detail, the condition for disturbance rejection will be discussed.

A. Disturbance Rejection Condition

It is not convienient for every system to neglect disturbance in sliding mode. From a very general viewpoint, there must exist a control $\boldsymbol{u}$ that assures the dynamics of the real system (1) to equal the dynamics of the nominal system (6) considered for optimization

$$\exists \boldsymbol{u} \in \mathbb{R}^m : \; \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{z}}, \boldsymbol{u}^*) \tag{7}$$

for any desired control $\boldsymbol{u}^*$. This is called the *disturbance rejection condition*. This condition depends neither on the selected switching manifold nor on the sliding mode controller, it rather provides a design rule for the placement of the system actuators. In case this condition is not satisfied and disturbance is significant, position of the actuators in the system should be changed or an additional actuator should be used to provide better controllability with respect to the influence of the disturbance.

B. First Step

In the first design step an optimal continuous feedback control law $\boldsymbol{u}^*(\boldsymbol{x})$ has to be found for the nominal system (6). The solution to the optimization problem

$$\underset{\boldsymbol{u}(t)}{\operatorname{argmin}}\, J(\boldsymbol{x}, \bar{\boldsymbol{z}}, \boldsymbol{u}, T) = \underset{\boldsymbol{u}(t)}{\operatorname{argmin}} \int_0^T \Pi(\boldsymbol{x}, \bar{\boldsymbol{z}}, \boldsymbol{u}) \mathrm{d}t \tag{8}$$

taking into account the equality constraint

$$\dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{z}}, \boldsymbol{u}) = \boldsymbol{0} \tag{9}$$

and the boundary conditions

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \text{ and } \boldsymbol{x}(T) = \boldsymbol{x}_T \tag{10}$$

is the control law $\boldsymbol{u}^*(\boldsymbol{x})$. The corresponding optimal system trajectory $\boldsymbol{x}^*(t)$ of the nominal system (6) then is the solution to the optimally controlled nominal system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{z}}, \boldsymbol{u}^*(\boldsymbol{x})) =: \boldsymbol{f}^*(\boldsymbol{x}) \,. \tag{11}$$

Applied to a linear system with $\Pi(\boldsymbol{x}, \bar{\boldsymbol{z}}, \boldsymbol{u})$ in quadratic form, this results in the well known Riccati matrix differential equation. The solution to this equation is the LQ-controller.

C. Second Step

In the second step a switching function $\boldsymbol{s}^*(\boldsymbol{x})$ has to be found based on the optimal continuous control law $\boldsymbol{u}^*$. Condition (3) provides the equation

$$\dot{\boldsymbol{s}}(\boldsymbol{x}) = \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} + \frac{\partial \boldsymbol{s}}{\partial t} = \boldsymbol{0},\, \boldsymbol{s} \in \mathbb{R}^m \,. \tag{12}$$

Since the system should follow the optimal system dynamics $\dot{\boldsymbol{x}} = \boldsymbol{f}^*(\boldsymbol{x})$, this condition leads to

$$\begin{aligned} \dot{\boldsymbol{s}}(\boldsymbol{x}) = \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) + \frac{\partial \boldsymbol{s}}{\partial t} \overset{!}{=} \boldsymbol{0},\, \forall \boldsymbol{x} : \; \dot{\boldsymbol{x}} = \boldsymbol{f}^*(\boldsymbol{x}) \\ \Rightarrow \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{x}} \boldsymbol{f}^*(\boldsymbol{x}) + \frac{\partial \boldsymbol{s}}{\partial t} \overset{!}{=} \boldsymbol{0} \end{aligned} \tag{13}$$

The m independent solutions $\boldsymbol{s}^*(\boldsymbol{x})$ of this set of partial differential equations contain m constants. They are determined by the m equations of condition (2)

$$\boldsymbol{s}^*(\boldsymbol{x}) \overset{!}{=} \boldsymbol{0},\, \forall \boldsymbol{x} : \; \dot{\boldsymbol{x}} = \boldsymbol{f}^*(\boldsymbol{x}) \,. \tag{14}$$

Once sliding mode occurs, condition (13) assures that $\boldsymbol{s}^*$ has a constant value. Hence, if condition (14) is satisfied for an arbitrary point of the optimal trajectory $\boldsymbol{x}^*(t)$, then $\boldsymbol{s}^*$ is forced to zero for the entire trajectory. This means that the switching function $\boldsymbol{s}^*$, that fulfills the above conditions (13) and (14), forces the system to sliding mode starting from initial state $\boldsymbol{x}_0$ to target state $\boldsymbol{x}_T$ without reaching phase. This again means the sliding mode controller is only responsible for keeping the system state on the switching manifold.

In order to find $\boldsymbol{s}^*$ the relation between the optimal system trajectory and the conditions for the switching function $\boldsymbol{s}^*$ in sliding mode is considered. The system trajectory in general is the solution to the differential equation of the system (1) and may be written in the following implicit form

$$\boldsymbol{r}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \boldsymbol{0},\, \boldsymbol{r} \in \mathbb{R}^n,\, \forall \boldsymbol{x} : \; \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \,. \tag{15}$$

Equation (15) determines a set of spatial curves with parameters $\boldsymbol{z}$ and $\boldsymbol{u}$. The points $\boldsymbol{x}$ that solve (1) lie on this set of curves. Any explicit solution may also be written in this form. If time t can be eliminated, the implicit trajectory will be of dimension $\mathbb{R}^{n-1}$ instead of $\mathbb{R}^n$. The optimal system trajectory in implicit form is the solution to the differential equation of the optimally controlled nominal system (11)

$$\boldsymbol{r}^*(\boldsymbol{x}) = \boldsymbol{0},\ \forall \boldsymbol{x} :\ \dot{\boldsymbol{x}} = \boldsymbol{f}^*(\boldsymbol{x})\,. \tag{16}$$

With $\boldsymbol{z} = \bar{\boldsymbol{z}}$ and $\boldsymbol{u} = \boldsymbol{u}^*(\boldsymbol{x})$, all parameters are assessed and (16) defines a single spatial curve. The points $\boldsymbol{x}^*$ that solve (11) lie on this curve. From (16) it follows that

$$\begin{aligned} \dot{\boldsymbol{r}}^*(\boldsymbol{x}) = \frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \frac{\partial \boldsymbol{r}^*}{\partial t} = \boldsymbol{0},\ \forall \boldsymbol{x} :\ \dot{\boldsymbol{x}} = \boldsymbol{f}^*(\boldsymbol{x}) \\ \Rightarrow \qquad \frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{x}} \boldsymbol{f}^*(\boldsymbol{x}) + \frac{\partial \boldsymbol{r}^*}{\partial t} = \boldsymbol{0}\,. \end{aligned} \tag{17}$$

Equation (17) determines, that the state remains on the trajectory curve and moves along it without leaving it.

The similarity between conditions (14) and (13) and the trajectory equations (16) and (17) is apparent. Since $\boldsymbol{r}^*$ already fulfills the necessary conditions for $\boldsymbol{s}^*$ but in $\mathbb{R}^n$ instead of $\mathbb{R}^m$, now it is possible to choose the switching function as a function of $\boldsymbol{r}^*$

$$\boldsymbol{s}^* := \boldsymbol{P}\boldsymbol{r}^* \tag{18}$$

with $\boldsymbol{P}$ being a constant projection matrix of dimension $\mathbb{R}^{m\times n}$. For systems that are linear in control and that fulfill the disturbance rejection condition (7) and the base condition

$$\det\left(\boldsymbol{P}\frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right) \neq 0 \tag{19}$$

the correspondence

$$\boldsymbol{s}^* = \boldsymbol{0} \Leftrightarrow \boldsymbol{r}^* = \boldsymbol{0} \tag{20}$$

is valid.

D. *Proof of correspondence (20)*

Since $\boldsymbol{s}^* \in \mathbb{R}^m$, $\boldsymbol{r}^* \in \mathbb{R}^n$ and $m \leq n$, the necessary condition

$$\boldsymbol{r}^* = \boldsymbol{0} \Rightarrow \boldsymbol{s}^* = \boldsymbol{0} \tag{21}$$

is true.

The proof of the sufficient condition relies on the linearization of the system function around the optimal trajectory. It can be proven that once this optimal trajectory is left the switching manifold is also left.

The linearization of a continuous function $\boldsymbol{f}(\boldsymbol{x})$

$$\boldsymbol{f}(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\mathrm{d}\boldsymbol{x} \tag{22}$$

is exact if all derivations $\mathrm{d}x_i$ are infinitesimally small or if $\frac{\partial \boldsymbol{f}}{\partial x_j}$ is independent of x_j and all other $\mathrm{d}x_{i\neq j}$ are infinitesimally small. In the second case $\boldsymbol{f}$ is linear with respect to x_j and the linearization is exact even if the derivation $\mathrm{d}x_j$ is not infinitesimally small.

In sliding mode control, linearization of the system function (1) with respect to $\boldsymbol{z}$ and $\boldsymbol{u}$

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z} + \mathrm{d}\boldsymbol{z}, \boldsymbol{u} + \mathrm{d}\boldsymbol{u}) \approx \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}\mathrm{d}\boldsymbol{z} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\mathrm{d}\boldsymbol{u} \tag{23}$$

is therefore exact, if $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}$ does not depend on $\boldsymbol{u}$ since the control input in sliding mode is a discontinuous function and therefore the derivation $\mathrm{d}\boldsymbol{u}$ is in general not infinitesimally small even for small derivations of the system trajectory. Thus, for the exact linearization of the system function in sliding mode to be exact, the system has to be linear in control. The same applies to the disturbance, if it is discontinuous as well, like noise for example, then the system function must be linear with respect to that variable as well.

With the exact linearization (23) the disturbance rejection condition (7) with $\bar{\boldsymbol{z}} := \boldsymbol{z} + \mathrm{d}\boldsymbol{z}$ and $\boldsymbol{u}^* := \boldsymbol{u} + \mathrm{d}\boldsymbol{u}$ takes the form

$$\begin{aligned} \exists \boldsymbol{u} \in \mathbb{R}^m :\ \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) &= \boldsymbol{f}(\boldsymbol{x},\bar{\boldsymbol{z}},\boldsymbol{u}^*) \\ &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}\mathrm{d}\boldsymbol{z} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\mathrm{d}\boldsymbol{u} \end{aligned} \tag{24}$$

and therefore simplifies to

$$\exists \mathrm{d}\boldsymbol{u} \in \mathbb{R}^m :\ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}\mathrm{d}\boldsymbol{z} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\mathrm{d}\boldsymbol{u} = \boldsymbol{0}\,. \tag{25}$$

This can only be fulfilled if

$$\exists \mathrm{d}\boldsymbol{u}_z \in \mathbb{R}^m :\ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}\mathrm{d}\boldsymbol{z} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\mathrm{d}\boldsymbol{u}_z \tag{26}$$

which is called the *matching condition*. [2]

Linearization of the system function (1) around the optimal trajectory defined by $\boldsymbol{z} = \bar{\boldsymbol{z}}$ and $\boldsymbol{u} = \boldsymbol{u}^*$ gives

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \underbrace{\boldsymbol{f}(\boldsymbol{x},\bar{\boldsymbol{z}},\boldsymbol{u}^*)}_{\boldsymbol{f}^*(\boldsymbol{x})} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}(\boldsymbol{z} - \bar{\boldsymbol{z}}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{u} - \boldsymbol{u}^*) \tag{27}$$

and with the disturbance rejection condition (26) fulfilled (27) leads to

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) &= \boldsymbol{f}^*(\boldsymbol{x}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{u}_z - \boldsymbol{u}_{\bar{z}}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{u} - \boldsymbol{u}^*) \\ &= \boldsymbol{f}^*(\boldsymbol{x}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\boldsymbol{v} \end{aligned} \tag{28}$$

with $\boldsymbol{v} := \boldsymbol{u}_z - \boldsymbol{u}_{\bar{z}} + \boldsymbol{u} - \boldsymbol{u}^*$. The derivation can be calculated as

$$\dot{\boldsymbol{r}}^* = \frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \frac{\partial \boldsymbol{r}^*}{\partial t}\,, \tag{29}$$

with (17) and (28) this leads to

$$\dot{\boldsymbol{r}}^* = \frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\boldsymbol{v}\,. \tag{30}$$

From the presumption $\boldsymbol{s}^* = \boldsymbol{0}$ it follows that $\dot{\boldsymbol{s}}^* = \boldsymbol{0}$ and with (18) and (29) this leads to

$$\dot{\boldsymbol{s}}^* = \boldsymbol{P}\dot{\boldsymbol{r}}^* = \boldsymbol{P}\left(\frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \frac{\partial \boldsymbol{r}^*}{\partial t}\right) = \boldsymbol{0}\,, \tag{31}$$

with (17) and (28) follows

$$\dot{\boldsymbol{s}}^* = \boldsymbol{P}\frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\boldsymbol{v} = \boldsymbol{0}\,. \tag{32}$$

If the determinant fullfills the base condition (19) then $\boldsymbol{v}$ must be the zero vector in order to fulfill $\dot{\boldsymbol{s}}^* = \boldsymbol{0}$ in (32). Hence, $\boldsymbol{v} = \boldsymbol{0}$ and (30) leads to

$$\dot{\boldsymbol{r}}^* = \boldsymbol{0}\,. \tag{33}$$

Since the initial state $\boldsymbol{x}_0$ lies on $\boldsymbol{r}^* = 0$

$$\boldsymbol{r}_0^* = \boldsymbol{0} \tag{34}$$

and hence

$$\boldsymbol{r}^* = \boldsymbol{r}_0^* + \int_0^t \dot{\boldsymbol{r}}^* \mathrm{d}t = \boldsymbol{r}_0^* = \boldsymbol{0},\, \forall t \geq 0\,, \tag{35}$$

the optimal trajectory is never left and the sufficient condition

$$\boldsymbol{s}^* = \boldsymbol{0} \Rightarrow \boldsymbol{r}^* = \boldsymbol{0} \tag{36}$$

is true.

E. Third Step

In the third design step, a switching controller $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{s}^*)$ has to be found which enforces sliding mode in the manifold $\boldsymbol{s}^* = \boldsymbol{0}$. It is a controller that only satisfies the reaching condition (5). No optimization with respect to a cost functional $J(\boldsymbol{x}, \bar{\boldsymbol{z}}, \boldsymbol{u}, T)$ has to be made, since no reaching phase, where the system trajectory would depend on the controller, appears in the system. Some widely used sliding mode controllers may be found in the literature [2].

IV. MINIMIZING DISSIPATIVE POWER LOSS SLIDING MODE CONTROL OF A SYNCHRONOUS MOTOR

This section focusses on the application of the design method to a real system. A three-phase permanent magnet synchronous motor, whose angular position has to be controlled, is considered. Optimization has to be performed in order to minimize the dissipative power loss during operation. The disturbance is the variable load torque. It enables a very simple evaluation of the optimality and disturbance rejection capability of the controlled system.

A. Machine Models

The synchronous motor is modeled in rotating rotor coordinates $\{d, q\}$. Unlike in stator coordinates, where the signals have to be modulated on a sine wave in order to propel the machine, the waveform of all variables in rotor coordinates is unconstrained and the modulation is done implicitly during transformation from rotor to stator coordinates. The model is given by the differential equations

$$u_d = Ri_d + L\dot{i}_d - p\omega L i_q \tag{37}$$

$$u_q = Ri_q + L\dot{i}_q + p\omega L i_d + \frac{2}{3}k\omega \tag{38}$$

$$ki_q = J\dot{\omega} + f\omega + M \tag{39}$$

with the following variables and parameters:

u_d, u_q:	direct and quadrature voltage,
i_d, i_q:	direct and quadrature current,
ω:	motor shaft angular velocity,
M:	disturbing external torque,
R, L:	winding resistance and inductivity,
J:	rotor and shaft inertia,
p:	number of permanent magnet pole pair,
k:	machine constant and
f:	viscous bearing friction coefficient.

System (37)-(39) does not fulfill the disturbance rejection condition (7), because the dynamic equation (39) does depend on the disturbance M but not on the control $\boldsymbol{u} = [u_d \, u_q]^T$. Introduction of an angular acceleration γ solves this problem. The dynamic system

$$\begin{aligned} \dot{\omega} &= \gamma \\ \dot{\gamma} &= -a\omega - b\gamma - \frac{pk}{J}\omega i_d - \frac{R}{LJ}M - \frac{1}{J}\dot{M} + \frac{k}{LJ}u_q \\ \dot{i}_d &= -\frac{R}{L}i_d + \frac{pf}{k}\omega^2 + \frac{pJ}{k}\omega\gamma + \frac{p}{k}\omega M + \frac{1}{L}u_d \end{aligned} \tag{40}$$

with

$$\begin{aligned} a &= \left(\frac{Rf}{LJ} + \frac{2k^2}{3LJ}\right) \\ b &= \left(\frac{R}{L} + \frac{f}{J}\right) \end{aligned} \tag{41}$$

is obtained, which fulfills the disturbance rejection condition (7). Since the angular position of the motor has to be controlled, the following supplementary equation has to be added to the system

$$\dot{\theta} = \omega \tag{42}$$

with θ denoting the angular position of the motor shaft.

The design method that was introduced in section 3 is now applied to the system (40) of a synchronous motor.

B. First Step

The control objective is the minimization of the dissipative power loss Π of the synchronous machine

$$\Pi := \frac{3}{2}(R + r)(i_d^2 + i_q^2) + f\omega^2 \tag{43}$$

during operation. Parameter r represents the internal resistance of the switching semiconductors connected in series to the winding resistance R on each phase.[1]

In order to be able to carry out the optimization analytically, a simplified motor model is used. This model is derived from the original motor model equations (37), (38), (39) and (42) by neglecting the transients of the electrical components of the state which have very small time constants compared to the mechanical components.[2] This is done by setting

$$\dot{i}_d = 0 \text{ and } \dot{i}_q = 0\,. \tag{44}$$

Furthermore, equation (39) shows that the direct current i_d does not contribute to the motor torque ki_q.[3] The optimization task can therefore be further simplified by fixing i_d to an optimal value i_d^* using control input u_d. Considering the function (43), the optimal value of i_d is

$$i_d^* = 0\,. \tag{45}$$

With (44) and (45), the model equations (37), (38), (39) and (42) and assigning $\boldsymbol{u} := u_q$ the reduced state space

---

[1] The factor $\frac{3}{2}$ is a result of the transformation of the three dependent phase currents i_a, i_b and i_c into the two independent currents i_d and i_q.

[2] This simplification is a special case of the theory of singular perturbation.

[3] In fact, this is an approximation. The real electromagnetic torque is calculated by $M_{em} = ki_q + M_r$. The reluctant torque $M_r = \frac{3}{2}p(L_d - L_q)i_d i_q$ can be neglected for machines with surface mounted magnets like the test machine, because $L_d = L_q =: L$ for these machines. For more details see [9].

model of the synchronous motor

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{bmatrix}\dot{\theta}\\ \dot{\omega}\end{bmatrix}\\ &= \begin{bmatrix}\omega\\ -\frac{f}{J}\omega + \frac{k}{RJ}\left(u_q - \frac{2}{3}k\omega\right) - \frac{1}{J}M\end{bmatrix}\\ &= \begin{bmatrix}f_1(\boldsymbol{x})\\ f_2(\boldsymbol{x}, M, \boldsymbol{u})\end{bmatrix} = \boldsymbol{f}(\boldsymbol{x}, M, \boldsymbol{u})\end{aligned} \tag{46}$$

is obtained. The nominal value of the disturbance M is taken to be zero

$$\bar{M} = 0\,. \tag{47}$$

With θ_0 and θ_T being initial and target angular positions respectively and T being the selectable transition time from initial to target position, the optimization problem results in a problem of finding

$$\underset{\boldsymbol{u}(t)}{\operatorname{argmin}} J(\boldsymbol{x}, \bar{M}, \boldsymbol{u}, T) = \underset{\boldsymbol{u}(t)}{\operatorname{argmin}} \int_0^T \Pi(\boldsymbol{x}, \bar{M}, \boldsymbol{u})\mathrm{d}t \tag{48}$$

taking into account the equality constraint

$$\dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}, \bar{M}, \boldsymbol{u}) = \boldsymbol{0} \tag{49}$$

and the boundary conditions

$$\boldsymbol{x}(0) = \begin{bmatrix}\theta_0\\ 0\end{bmatrix} \text{ and } \boldsymbol{x}(T) = \begin{bmatrix}\theta_T\\ 0\end{bmatrix}\,. \tag{50}$$

Its solution leads to the optimal feedback control law

$$\theta^*(t) = \theta_0 + K\left[\sinh(qt) - \frac{D}{S}[\cosh(qt) - 1] - qt\right] \tag{51}$$

with

$$\begin{aligned}K &= \frac{S(\theta_T - \theta_0)}{2D - qTS}\\ D &= \cosh(qT) - 1\\ S &= \sinh(qT)\\ q &= \sqrt{\frac{f^2}{J^2} + \frac{2fk^2}{3(R+r)J^2}}\,.\end{aligned} \tag{52}$$

C. Second Step

Optimization has been carried out using a simplified system (46). The current changes slowly using a continuous controller and neglecting its fast transients seems to be valid. However the switching function for the sliding mode controller must take into account real system dynamics with disturbance rejection capability as given in (40) along with (42). The state vector is therefore $\boldsymbol{x} = [\theta\,\omega\,\gamma\,i_d]^T$ and the control input vector $\boldsymbol{u} = [u_d\,u_q]^T$. The disturbance vector $\boldsymbol{z}$ contains the variable load torque M, its derivative $\dot{M}$ and various uncertain parameters, above all the shaft inertia J. One possibility of defining the optimally controlled trajectory in implicit form is then given by

$$\boldsymbol{r}^* = \boldsymbol{x} - \boldsymbol{x}^* = \begin{bmatrix}\theta - \theta^*\\ \omega - \omega^*\\ \gamma - \gamma^*\\ i_d - i_d^*\end{bmatrix} = \boldsymbol{0}\,. \tag{53}$$

With θ^* given by (51), $\omega^* = \dot{\theta}^*$, $\gamma^* = \ddot{\theta}^*$ and i_d^* given by (45). According to (18), the two dimensional switching function can be found as (18) with the projection matrix $\boldsymbol{P}$ respecting the base condition (19)

$$\det\left(\boldsymbol{P}\frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right) \overset{!}{\neq} 0 \tag{54}$$

with

$$\begin{aligned}\boldsymbol{P} &= \begin{bmatrix}p_{11} & p_{12} & p_{13} & p_{14}\\ p_{21} & p_{22} & p_{23} & p_{24}\end{bmatrix}\\ \frac{\partial \boldsymbol{r}^*}{\partial \boldsymbol{x}} &= \boldsymbol{1}\\ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} &= \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & \frac{k}{LJ}\\ \frac{1}{L} & 0\end{bmatrix}\,.\end{aligned} \tag{55}$$

The system above leads to

$$\Leftrightarrow \quad \det\left(\begin{bmatrix}p_{14}\frac{1}{L} & p_{13}\frac{k}{LJ}\\ p_{24}\frac{1}{L} & p_{23}\frac{k}{LJ}\end{bmatrix}\right) \overset{!}{\neq} 0 \tag{56}$$

$$\Leftrightarrow \quad p_{14}p_{23} - p_{13}p_{24} \overset{!}{\neq} 0\,. \tag{57}$$

This condition allows to set

$$p_{11} = p_{12} = p_{13} = p_{24} = 0 \tag{58}$$

which separates the state variable i_d controlled by u_d from state variable γ controlled by u_q, ensuring the presupposition (45) that was made in order to simplify the optimization. Hence, the switching function in $\{d, q\}$ coordinates simplifies to

$$\boldsymbol{s}^* = \begin{bmatrix}p_{14}(i_d - i_d^*)\\ p_{21}(\theta - \theta^*) + p_{22}(\omega - \omega^*) + p_{23}(\gamma - \gamma^*)\end{bmatrix} \tag{59}$$

with θ^* given by (51), $\omega^* = \dot{\theta}^*$, $\gamma^* = \ddot{\theta}^*$ and i_d^* given by (45).

D. Third Step

The switching function (59) has now been generated in rotor coordinates $\{d, q\}$. Switching however takes place in stator $\{1, 2, 3\}$ coordinates, with the motor phase voltages taking values from the limited set $\{-u_0, u_0\}$. The switching control law must therefore be of the form

$$\boldsymbol{u}_{123} = U \operatorname{sign} \boldsymbol{s}_{123} \tag{60}$$

with $|U| = u_0$ and $\boldsymbol{s}_{123} \in \mathbb{R}^3$ being the transformation of (59) in stator coordinates

$$\boldsymbol{s}_{123} = \begin{bmatrix}s_d\cos(p\theta) - s_q\sin(p\theta)\\ s_d\cos\left(p\theta - \frac{2}{3}\pi\right) - s_q\sin\left(p\theta - \frac{2}{3}\pi\right)\\ s_d\cos\left(p\theta - \frac{4}{3}\pi\right) - s_q\sin\left(p\theta - \frac{4}{3}\pi\right)\end{bmatrix} \tag{61}$$

with $[s_d\,s_q]^T = \boldsymbol{s}^*$ from (59). It can be proven by applying the reaching condition (5) on Lyapunov function

$$V = \frac{1}{2}\boldsymbol{s}_{123}^T\boldsymbol{s}_{123} \tag{62}$$

so that

$$U = -u_0 \tag{63}$$

and a high enough link voltage u_0, the switching controller (60) enforces the system to sliding mode in the manifold $\boldsymbol{s}_{123} = \boldsymbol{0}$. For the detailed proof see [2].

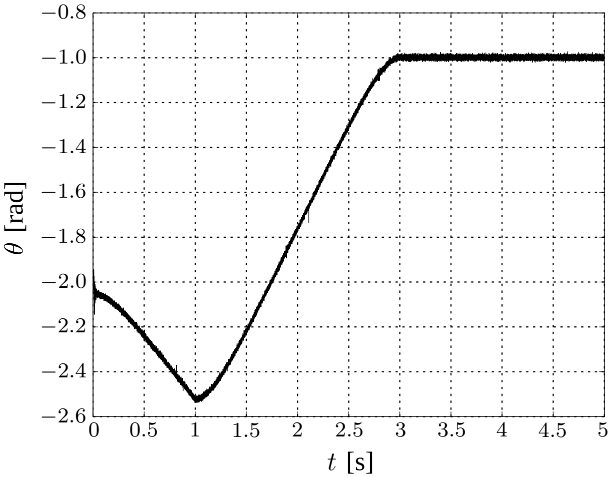

Fig. 1. Experimental result: Angular position θ when cost functional minimizing sliding mode control for $T = 2\,\mathrm{s}$ is applied. Control starts at time $t = 0\,\mathrm{s}$ with the target angle set to $\theta_T = -3\,\mathrm{rad}$. At $t = 1\,\mathrm{s}$ the target angle is changed to $\theta_T = -1\,\mathrm{rad}$.

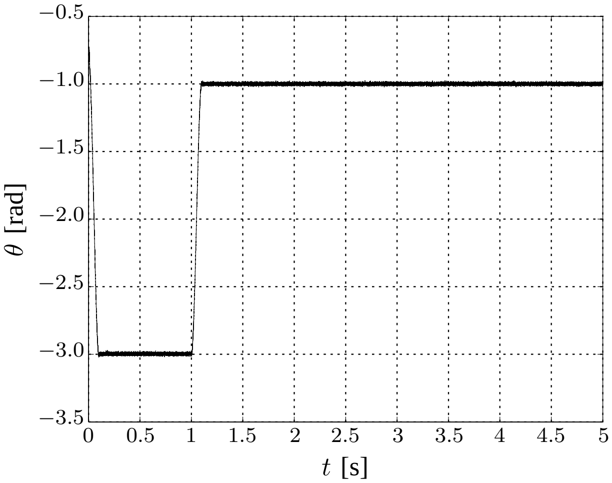

Fig. 2. Experimental result: Angular position θ when cost functional minimizing sliding mode control for $T = 0.1\,\mathrm{s}$ is applied. Control starts at time $t = 0\,\mathrm{s}$ with the target angle set to $\theta_T = -3\,\mathrm{rad}$. At $t = 1\,\mathrm{s}$ the target angle is changed to $\theta_T = -1\,\mathrm{rad}$.

E. Practical Application

The arrangement of the test bench to verify theoretical results consists of the motor with a potentiometer for position θ measurement attached to the shaft. The three phase voltages of the motor are switched by a dedicated three phase motor driving circuit, according to the sliding mode control law given in (60) with amplitude (63) and switching function (61). This control law is implemented on a computer that calculates the optimal trajectories of θ^*, ω^*, γ^* and i_d^* and observes the state variables ω, γ and i_d based on the position measurement. The last remaining variables in the control law are the elements p_{ij} of the projection matrix $\boldsymbol{P}$ in (59).

However, it has been stated in the previous section, that the projection matrix $\boldsymbol{P}$ does not influence the trajectory as long as it fulfills the base condition (57) which after (58) simplifies to

$$p_{14}p_{23} \overset{!}{\neq} 0\,. \qquad (64)$$

According to this condition, the remaining freedom of parameters of $\boldsymbol{P}$ in (59) can be used to normalize the noise levels between θ, ω and γ.[4] This is of particular importance since in the experimental setup speed ω and acceleration γ can not be measured and have to be observed.

Figures 1 and 2 show the results of the angular position θ after two runs of the same motor[5] but with the transition time T set to different values. Figure 1 shows the optimal trajectory due to a longer transition time better whereas figure 2 demonstrates the performance of the controlled motor. Since the disturbance rejection condition is fulfilled, there is no visible difference in the resulting trajectories with disturbance applied, as long as the disturbing torque does not surpass the maximal torque the motor can handle.

V. CONCLUSION

In this paper, a straightforward design method for a sliding mode controller that provides optimal system dynamics with respect to a given cost functional has been proposed. In sliding mode the system dynamics follow an optimal trajectory which is calculated based on well known methods of constrained continuous dynamic optimization. The developed three-step design process proved working on a test system, a three-phase permanent magnet synchronous motor. Despite the application of external disturbing torques, the motor reaches a target position on an optimal path, minimizing its dissipative power losses on the way.

VI. ACKNOWLEDGEMENT

Thanks to the Alexander von Humboldt Foundation for the support of this work.

REFERENCES

[1] V. Utkin. Variable Structure Systems with Sliding Modes, IEEE Transactions on Automatic Control, 22(2): 212–222, April 1977.

[2] V. Utkin, J. Guldner, J. Shi. Sliding Mode Control in Electromechanical Systems. The Taylor & Francis System and Control Book Series, 1999.

[3] H. Hebisch. Grundlagen der Sliding-Mode-Regelung. Forschungsbericht Nr. 15/95, Gerhard-Mercator-Universität, GH Duisburg, Institut für Mess-, Steuer- und Regelungstechnik, 1995.

[4] A. Ferrara, V. I. Utkin. Sliding Mode Optimization in Dynamic LTI Systems. Journal of Optimization Theory and Applications, 115(3): 727–740, Plenum Publishing Corporation, December 2002.

[5] K. D. Young, Ü. Özgüner. Sliding Mode Design for Robust Linear Optimal Control. Automatica, 33(7): 1313–1323, Elsevier Science Ltd., 1997.

[6] J. L. Crassidis, S. R. Vadali, F. L. Markley. Optimal Tracking of Spacecraft using Variable-Structure Control. Proceedings of the Flight Mechanics / Estimation Theory Symposium, NASA-CP 1999-209235: 201–214, NASA-Goddard Space Flight Center, May 1999.

[7] C.-L. Chen, M.-H. Chang. Optimal Design of Fuzzy Sliding-Mode Control: A Comparative Study. Fuzzy Sets and Systems, 93: 37–48, Elsevier Science B. V., 1998.

[8] M. Papageorgiou. Optimierung. R. Oldenbourg Verlag München Wien, 1991.

[9] S. Andersson. Optimization of a Servo Motor for an Industrial Robot Application. ISBN 91-88934-15-2, Lund 2000.

---

[4] A sliding mode controller rejects disturbance to the system but not measurement noise

[5] A Maxon EC45 flat motor was used