Performance-oriented Communication Topology Design for Large-scale Interconnected Systems

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***Abstract*— Communication networks provide a larger flexibility with respect to the control design of large-scale interconnected systems by allowing the information exchange between the local controllers of the subsystems. This paper introduces an approach for the design of the communication topology for the distributed control of large-scale interconnected systems in order to optimize the whole system's performance. First, a decentralized control law that stabilizes the overall interconnected systems is designed. Then the performance is improved by designing the distributed control law, i.e., allowing the controller of the subsystems to exchange information. As a novelty in this paper, the design of the communication topology between the controllers is considered. The problem is formulated as a mixed-integer optimization problem. Furthermore, it is shown that for a certain class of systems, the optimization problem can be reformulated resulting in a more explicit solution. In addition, it is shown that the proposed strategy also guarantees the stability of the whole system under the permanent communication link's failure. The results are validated through simulations.**

I. Introduction

The design of control algorithms for complex dynamical systems has become a vibrant part of research due to the wide applicability and impact with applications ranging from smart power grids, water distribution and traffic systems to large arrays of micro-electro-mechanical systems (MEMS), formation of vehicles, and sensor-actuator networks.

The key challenge for the control of large-scale dynamical systems is the complexity of the overall system in terms of the number of subsystems and their interconnections. First results addressing the complexity of large-scale systems have been achieved within the decentralized control framework developed since the seventies, see, e.g., [1] for a nice overview. Decentralized control algorithms may ensure the stability of the whole system. However, the performance is in general degraded compared to centralized control approaches as only the local subsystem information is used for the control.

Digital communication networks allow the communication between the subsystems and thereby provide a larger flexibility with respect to the control design: Instead of only local subsystem information also neighboring subsystems' states can be used for the control. These novel approaches are also known under the notion of *distributed control* [2]. As a result, typically a better performance is achieved compared to the standard decentralized approach [3]–[5]. Another merit of using information from the neighboring subsystems is that it can be used to stabilize the interconnected systems when no stabilizing decentralized control law exists, for example in the presence of decentralized fixed modes (DFM) in the system [6].

Most of the works in the known literature for distributed systems deal with the problem of stabilization of large-scale systems, e.g., [7]–[9]. An approach which combines distributed control and optimization can be found in [10], [11] where the idea of dual decomposition is used for decomposition and distributed optimization of feedback systems. Specifically, distributed optimization is used to iteratively update local controllers of a distributed system based on a gradient approach. The combination of both methods, i.e., optimization and feedback control thereby results in a better system's performance and a guaranteed stability. The works mentioned above assume that the communication topology is given. The introduction of a communication network, however, provides an additional degree of freedom for the structural design of the distributed controller in terms of the communication topology. The problem of designing communication topology has been studied in the area of sensor networks where the goal is to minimize the sensors' power consumption while maintaining the connectivity of the network, e.g., in [12]. The communication topology also plays an important role in the performance and stability of the controlled overall system as indicated by several results especially in the area of multi-agent systems. For example, in [13] it is shown that the connectivity of the agents could improve the convergence rate of the whole system to attain a desired behavior while decreasing the robustness of the systems to the time delay. The authors in [14] present the design of the topology of a random sensor network to maximize the convergence rate of the consensus algorithm. Moreover, the topology design to optimize a certain H_2 performance is investigated in [15]. The authors in [16] consider the synchronization of identical networked systems by introducing a distributed controller where the controller gain and structure are optimized subject to a given constraint. Nevertheless, the design of communication topology for a more general system is still an open problem.

In this paper, we consider the design of the communication topology between the controllers thereby exploring the additional degrees of freedom offered by the communication network. The goal is to improve the performance of the interconnected systems and in addition increase the robustness w.r.t. the permanent link failures. In order to achieve the goals, first we design the decentralized control to stabilize the overall interconnected systems. Then, we improve the

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performance of the systems by designing a controller which uses the state information from other subsystems under given communication network constraints. As a performance metric we consider the convergence rate. The approach also guarantees the stability, i.e., increases the robustness of the overall system, in the presence of permanent communication link failures.

The remainder of the paper is organized as follows. We first design the decentralized control law that stabilizes the interconnected system in Section III. The decentralized control design is formulated as an LMI problem. The design of the communication topology in order to improve the performance of the systems is considered in Section IV where the problem is formulated as a mixed-integer optimization problem. A more explicit formulation on the optimization are also provided for a certain class of systems. The proposed approach is demonstrated through a numerical example in Section V.

II. Problem Formulation

Consider an interconnected systems of N heterogenous linear time invariant (LTI) subsystems described by the following differential equations

$$\dot{x}_i = A_i x_i + \sum_{j\in N_i} A_{ij} x_j + B_i u_i, \tag{1}$$

where $i = 1,2,...,N$ denotes the $i-$th subsystem, $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^p$ are the state of subsystem i and the control input to subsystem i, and $A_i, A_{ij} \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times p}$. The term $\sum_{j\in N_i} A_{ij} x_j$ represents the physical interconnection between the subsystems where N_i is the set of subsystems to which subsystem i is physically connected. The interconnected systems described above can be found for example in power grids. Here we consider a state feedback controller for which the control law can be written as follows.

$$u_i = K_i x_i + \sum_{j\in G_i} K_{ij} x_j, \tag{2}$$

which is known as *distributed* control law since the controller for each subsystem does not only depend on its own states but also the states of the other subsystems. Here G_i represents a set of subsystems to which controller i could communicate, i.e., exchange information. The interconnected system with its distributed controller is illustrated in Fig. 1. If $K_{ij} = 0, \forall i$ and $\forall j \in G_i$, then the control law is called a *decentralized* control law.

The goal is to design the distributed control (2) such that the performance of the whole system is improved and the stability of the system is guaranteed. Moreover, as an additional goal, the robustness of the interconnected system w.r.t. the permanent link failures has to be guaranteed. In order to achieve the goals, first we design a decentralized control that guarantees the stability of the interconnected system. The performance of the systems is then improved by designing a distributed control law, i.e., the second term of (2). Furthermore, as one part of the innovation of this paper, the communication topology, i.e., $G_i, \forall i$ of the distributed control law is also a design parameter.

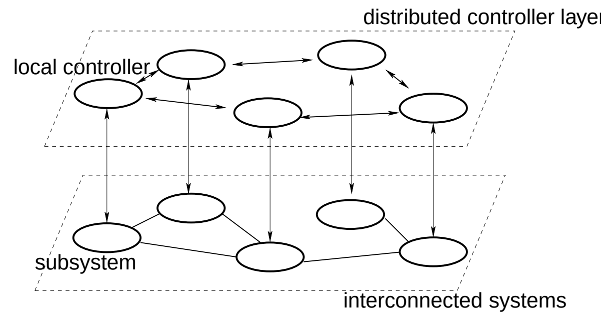

Fig. 1. Interconnected systems with distributed control architecture

III. Decentralized Control Design

In this section we design the decentralized control law that stabilizes the interconnected system (1). First, the stability of the following interconnected systems is investigated.

$$\dot{x}_i = A_i x_i + \sum_{j\in N_i} A_{ij} x_j. \tag{3}$$

Proposition 3.1: If there exists positive constants $\varepsilon_i > 0$ such that the following N linear matrix inequalities have symmetrically positive definite solutions P_i,

$$\begin{bmatrix} A_i Q_i + Q_i A_i^T + \frac{1}{\varepsilon_i}(\sum_{j\in N_i} A_{ij} A_{ij}^T) & Q_i \\ Q_i & -\frac{1}{\varepsilon_i (N-1)} I_n^{-1} \end{bmatrix} < 0, \tag{4}$$

where $Q_i = P_i^{-1}$ and I_n is the identity matrix with dimension of n, then the interconnected system (3) is asymptotically stable.

In order to prove the proposition, we need the following lemma.

Lemma 3.1: [17] Let X, Y be two matrices of appropriate dimension. Then, for any $\varepsilon > 0$, the following inequality holds.

$$X^T Y + Y^T X \leq \varepsilon X^T X + \frac{1}{\varepsilon} Y^T Y.$$

Proof: The proof is similar to the one in [18]. First, let us consider the following Lyapunov function

$$V(x) = \sum_{i=1}^{N} x_i^T P_i x_i.$$

Taking the derivative of $V(x)$ along (3) gives:

$$\dot{V}(x) = \sum_{i=1}^{N} \dot{x}_i^T P_i x_i + x_i^T P_i \dot{x}_i. \tag{5}$$

Substituting (3) into the above equation gives

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^{N} x_i^T (A_i^T P_i + P_i A_i) x_i + (\sum_{j\in N_i} A_{ij} x_j)^T P_i x_i \\ &\quad + x_i^T P_i \sum_{j\in N_i} A_{ij} x_j, \end{aligned}$$

and applying Lemma 1 yields

$$\begin{aligned} \dot{V}(x) &\leq \sum_{i=1}^{N} x_i^T \Big[A_i^T P_i + P_i A_i + \frac{1}{\varepsilon} P_i (\sum_{j\in N_i} A_{ij} A_{ij}^T) P_i \\ &\quad + \varepsilon (N-1) I_n \Big] x_i. \end{aligned} \tag{6}$$

If $\forall i, A_i^T P_i + P_i A_i + \frac{1}{\varepsilon} P_i (\sum_{j\in N_i} A_{ij} A_{ij}^T) P_i + \varepsilon (N-1) I_n < 0$, then $\dot{V}(x)$ is negative definite. Thus, the interconnected systems (1) is stable. Pre- and post-multiply (6) by P_i^{-1} respectively gives

$$A_i Q_i + Q_i A_i^T + \varepsilon_i Q_i (N-1) I_n Q_i + \frac{1}{\varepsilon_i} (\sum_{j\in N_i} A_{ij} A_{ij}^T) < 0. \quad (7)$$

Finally, applying Schur complement to (7) results in (4). ■

Next, we consider the decentralized controller synthesis for the interconnected system (1) with the control input

$$u_i = K_i x_i. \quad (8)$$

Let $A_i + B_i K_i = \bar{A}_i$ and we will have a similar expression as in (4) given by

$$\begin{bmatrix} \bar{A}_i Q_i + Q_i \bar{A}_i^T + \frac{1}{\varepsilon_i} (\sum_{j\in N_i} A_{ij} A_{ij}^T) & Q_i \\ Q_i & -\frac{1}{\varepsilon_i (N-1)} I_n^{-1} \end{bmatrix} < 0. \quad (9)$$

The difficulty in solving the feedback gain K_i in the matrix inequality (9) is that it involves the nonlinear terms, i.e., $\bar{A}_i Q_i$, thus it can not be considered as an LMI problem. However, by restricting the solution space of Q_i, the nonlinear terms can be eliminated and the LMI problem is recovered.

Proposition 3.2: If there exists positive constants $\varepsilon_i > 0$ such that the following N linear matrix inequalities have symmetric solutions $Q_i > 0$ and Y_i,

$$\begin{bmatrix} A_i Q_i + (A_i Q_i)^T + B_i Y_i + \\ +(B_i Y_i)^T + \frac{1}{\varepsilon_i} (\sum_{j\in N_i} A_{ij} A_{ij}^T) & Q_i \\ Q_i & -\frac{1}{\varepsilon_i (N-1)} I_n^{-1} \end{bmatrix} < 0,$$

where $Q_i = P_i^{-1}$, then the decentralized controller is given by $K_i = Y_i Q_i^{-1}$.

Remark 1: It is assumed that all the decentralized fixed modes, if any, are in the open left half plane. The problem with unstable DFMs is the subject of future works.

IV. COMMUNICATION TOPOLOGY DESIGN

In this section the design of the communication topology will be investigated.

A. Optimization formulation

Assume that the decentralized control law stabilizing the interconnected system (1), i.e., the first term of (2), has been designed. Next we will improve the performance of the systems for a certain performance metric by designing the distributed controller, i.e., the feedback gain and the communication topology under given communication constraint. Here the convergence rate of the overall system is considered as a performance metric.

The closed loop expression of the interconnected systems (1) with the decentralized control law can be written as

$$\dot{x} = Ax, \; x(t_0) = x_0, \quad (10)$$

where $x = [x_1, x_2, \cdots, x_N]^T$ and

$$A = \begin{bmatrix} \bar{A}_1 & A_{12} & \cdots & A_{1N} \\ A_{21} & \bar{A}_2 & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & \bar{A}_N \end{bmatrix} \in \mathbb{R}^{nN\times nN}.$$

It is well known that the solution of (10) is given by $x(t) = e^{A(t-t_0)} x_0$ and the state norm satisfies

$$\|x(t)\| \le e^{\mathrm{Re}\{\lambda_{\max}(t-t_0)\}} \|x_0\|, \forall t \ge t_0, \quad (11)$$

where $\mathrm{Re}\{\lambda_{\max}\}$ represents the real part of the largest eigenvalues of A. The matrix A can then be decomposed into

$$\begin{aligned} A &= \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{A}_N \end{bmatrix} + \begin{bmatrix} 0 & A_{12} & \cdots & A_{1N} \\ A_{21} & 0 & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & 0 \end{bmatrix}, \\ A &= A_{\mathrm{ind}} + A_{\mathrm{int}}, \end{aligned}$$

where A_{ind} represents the dynamics of the isolated subsystems while A_{int} represents the interconnections between the subsystems.

The objective is to improve the performance of the overall systems, i.e., increase the convergence rate by designing the second term of (2) given by the following controller

$$\bar{u}_i = \sum_{j\in G_i} d_{ij} K_{ij} x_j, \quad (12)$$

where $d_{ij} \in \{0,1\}$ is a binary number that shows the possibility to perform the state information exchange between controller i and j, i.e., $d_{ij} = 1$ means that a communication link is added between controllers i and j and vice versa.. The new closed loop expression of (1) with the addition of controller (12) is given by

$$\dot{x} = \bar{A} x, \; x(t_0) = x_0, \quad (13)$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{A}_N \end{bmatrix} + \begin{bmatrix} 0 & \bar{A}_{12} & \cdots & \bar{A}_{1N} \\ \bar{A}_{21} & 0 & \cdots & \bar{A}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{N1} & \bar{A}_{N2} & \cdots & 0 \end{bmatrix}, \\ \bar{A} &= A_{\mathrm{ind}} + A_{\mathrm{dist}}. \end{aligned} \quad (14)$$

The term $\bar{A}_{ij}$ is defined as $\bar{A}_{ij} = A_{ij} + B_i d_{ij} K_{ij}$. Furthermore, we assume that not arbitrary many links can be added, i.e., the number is limited by an upper bound induced by the communication constraint

$$\sum_{1\le i\le j\le N} \gamma_{ij} d_{ij} \le c, \quad (15)$$

where $c > 0$ is the total cost constraint on the communication network, and γ_{ij} represents a cost to establish a link between subsystem i and j. This cost is typically related to factors such as the distance between the subsystems.

The problem can then be formulated as finding the gain and topology of the distributed controller such that the convergence rate of the overall interconnected system is optimized under a given communication constraint. The distributed controller is given by the following proposition.

Proposition 4.1: Consider an interconnected systems (13). If there exists a solution of the optimization problem

$$\begin{aligned} \underset{K_{ij},d_{ij}}{\text{minimize}} \quad & \text{Re}\{\lambda_{\max}(\bar{A})\} \\ \text{subject to} \quad & \text{Re}\{\lambda_{\max}(\bar{A})\} < \text{Re}\{\lambda_{\max}(A)\}, \\ & \sum_{1\le i\le j\le N} \gamma_{ij} d_{ij} \le c, \\ & d_{ij} \in \{0,1\}, \end{aligned} \tag{16}$$

then the convergence rate of the interconnected system with the distributed control law (2) is higher than with the decentralized control law (8) and the whole system remains stable.

Proof: Here we prove that the system remains stable if a solution of the optimization problem (16) exists. Since the interconnected systems is stabilized by the decentralized controller, then $\text{Re}\{\lambda_{\max}(A)\} < 0$. This yields

$$\text{Re}\{\lambda_{\max}(\bar{A})\} < \text{Re}\{\lambda_{\max}(A)\} < 0,$$

i.e., the interconnected systems with the distributed controller is stable. Moreover, since $\text{Re}\{\lambda_{\max}(\bar{A})\} < \text{Re}\{\lambda_{\max}(A)\}$, the convergence rate of the interconnected system with the distributed controller is higher than with the decentralized controller. ■

Remark 2: The optimization problem (16) is a mixed integer optimization problem since it is solved with respect to both the feedback gain and the communication topology of the distributed controller. The optimization (16) can be solved using well-known techniques such as relaxation and decomposition techniques or cutting planes approaches [19].

Remark 3: The approach in this paper can also be extended to distributed tracking control by first designing the decentralized output tracking control using the same approach as done, e.g., in [20], and then improving the tracking speed by a distributed control law which can be derived by solving an optimization problem similar to (16).

The stability of the whole system is also guaranteed in the presence of permanent communication link's failure as stated in the following Corollary.

Corollary 4.1: The stability of the interconnected systems (13) is guaranteed under any combination of permanent communication link's failures.

Proof: Since the maximum and minimum value of $\text{Re}\{\lambda_{\max}(\bar{A})\}$ are given by Proposition 3.1 and 4.1 respectively, the maximum eigenvalue of the whole system under any combination of communication link's failures, i.e., $\text{Re}\{\lambda_{\max}(\tilde{A})\}$ will be $\text{Re}\{\lambda_{\max}(\bar{A})\} < \text{Re}\{\lambda_{\max}(\tilde{A})\} < \text{Re}\{\lambda_{\max}(A)\} < 0$. ■

B. Adding communication links is not always beneficial

In the subsequent subsections we aim at deriving a more explicit solution of the optimization problem (16) using the tools from matrix perturbation analysis.

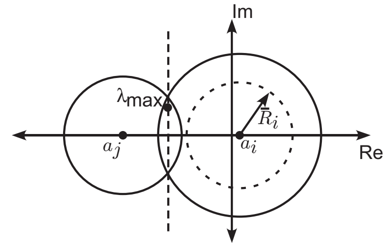

Fig. 2. Largest eigenvalue and Gershgorin discs

First, we will discuss that adding communication links is not always beneficial by finding a counterexample using eigenvalue perturbation theory. Before proceeding we review the well-known Gershgorin theorem. Consider therefore a complex $n \times n$ matrix with entries a_{ij}. For $i \in \{1,\cdots,n\}$ write $R_i = \sum_{j\neq i} |a_{ij}|$ and let $D(a_{ii}, R_i)$ be the disc centered at a_{ii} with radius R_i. Such a disc is called Gershgorin disc.

Theorem 4.1: [21] Every nonzero eigenvalue of A lies within at least one of the Gerhgorin discs $D(a_{ii}, R_i)$, i.e., every $\lambda(A)$ where $\lambda(.)$ represents eigenvalues of the corresponding matrix, satisfies $|\lambda - a_{ii}| \le R_i$ for some a_{ii}.

Theorem 4.2: [21] If the union of k discs is disjoint from the union of the other $n-k$ discs, then the former union contains exactly k and the latter $n-k$ eigenvalues of A.

Recall the equation (14) given by $\bar{A} = A_{\text{ind}} + A_{\text{dist}}$. The term A_{dist} can be seen as a perturbation working on the matrix A_{ind}. In order to show that adding communication links is not always beneficial, we need to show that the perturbation A_{dist} can possibly result in that the covergence rate of the overall system with matrix $\bar{A}$ is lower than with matrix $A = A_{\text{ind}} + A_{\text{int}}$, i.e., $\text{Re}\{\lambda_{\max}(\bar{A})\} > \text{Re}\{\lambda_{\max}(A)\}$. Consider a case where A is stable and its largest eigenvalue lies between the i-th and j-th Gershgorin disc centered at a_i and a_j, $a_i > a_j$ respectively as shown in Fig. 2. Now we would like to improve the performance by adding communication links between the subsystems under a given constraint on the total number of links. Here we assume directed communication links. Adding communication links into the i-th row means that only the i-th controller could receive information from the other controllers. Assume that the communication links are added only into the i-th row of matrix A_{dist}. From Theorem 4.1 and since the perturbation is only working on the non-diagonal elements of $\bar{A}$, the new Gershgorin discs have the same centers and the radius are scaled with the perturbation. Moreover, since the perturbations are only working on the i-th row of the matrix $\bar{A}$, only the radius of the i-th Gerhgorin disc changes. Thus, a combination of communication links together with the corresponding gain can be selected such that the i-th Gershgorin disc is disjoint from the other as illustrated in Fig. 2 and the new radius is less than the distance from its center to the largest eigenvalue of the non-perturbed matrix A, i.e., $\bar{R}_i \le \|a_i - \lambda_{max}(A)\|$. From Theorem 4.2, it is clear that the i-th Gershgorin disc has one eigenvalue inside and since $a_i > a_j > \cdots > a_N$, the corresponding eigenvalue is the largest eigenvalue of the perturbed matrix. This yields $\text{Re}\{\lambda_{\max}(\bar{A})\} > \text{Re}\{\lambda_{\max}(A)\}$.

C. Where to add the communication links?

In the previous subsection it has been discussed that the addition of communication links is not always beneficial since it could deteriorate the performance of the whole system. In this subsection we present a more explicit formulation on how to add the communication links, i.e., which controllers should communicate in order to improve the performance of the whole system for a given number of available communication links. In general, it is hard to derive an analytical solution to the optimization problem (16). In order to analyze the optimal topology design, we constrain ourselves for the remainder of this paper by the following assumptions.

- A1 The matrix A is a real symmetric matrix, i.e., $A^T = A$
- A2 The subsystems are scalar, i.e., $x_i \in \mathbb{R}$
- A3 The communication is bidirectional, i.e., $A_{\text{dist}}^T = A_{\text{dist}}$
- A4 The interconnected system with decentralized control law has equal diagonal elements, i.e., $\bar{A}_i = \cdots = \bar{A}_N$.

Since we are interested on the minimization of $\text{Re}\{\lambda_{max}(\bar{A}\}$, the eigenvalue sensitivity is analyzed in order to investigate how the structure of the distributed control law affects $\text{Re}\{\lambda_{max}(\bar{A}\}$. Eigenvalue sensitivity gives an insight on the behavior of the eigenvalues of a matrix when the matrix is perturbed, in our case, when the distributed control law is applied to the interconnected system. Moreover, the magnitude of the eigenvalue sensitivity informs about the size of the eigenvalue displacement in the complex plane [22]. Assume that the matrix A is perturbed by a matrix $M = [m_{ij}]$ with identical entries equal to m, i.e., $m_{ij} = m$, $\forall i, j$. Then the change of each eigenvalue of the matrix A is given by [23]

$$\frac{\partial \lambda_i}{\partial m} = \frac{v_i^T \frac{\partial \bar{A}}{\partial m} u_i}{v_i^T u_i}, \tag{17}$$

where $\bar{A} = A + M$, v_i and u_i are the right and left eigenvector of the matrix A corresponding to the eigenvalue λ_i.

Next we present the results on where to add the communication links and how to choose the feedback gain. For the simplicity of analysis and clarity of the result, it is assumed that $\gamma_{ij} = 1, \forall i, j$ and $c = 1$, i.e., only one link is allowed to be added. The perturbation matrix M can be seen as a distributed controller given by $A_{\text{dist}} = [K_{ij}]$ where $K_{ij} = K$. Let $v_r = [v_{r_1}, \cdots, v_{r_N}]^T$ be the eigenvector corresponding to $\lambda_{\max}(A)$.

Proposition 4.2: Consider an interconnected systems (13) under assumption A1-A4. The optimization problem (16) is minimized by adding a communication link between the $i-$th and $j-$th controller that solves the following optimization problem

$$\underset{i,j}{\text{maximize}} \ |v_{r_i} v_{r_j}| \tag{18}$$

and the sign of the feedback gain is chosen according to the following rules.

$$K = \begin{cases} > 0 & \text{if } \text{sign}(v_{r_i}.v_{r_j}) < 0 \\ < 0 & \text{if } \text{sign}(v_{r_i}.v_{r_j}) > 0. \end{cases} \tag{19}$$

Proof: In order to find the optimal communication topology that minimizes $\lambda_{\max}$, first we need to investigate the condition that results in $\lambda_{\max}(\bar{A}) < \lambda_{\max}(A)$, i.e., $\frac{\partial \lambda_{\max}}{\partial K} < 0$. Then we find the structure of perturbation, i.e., the distributed controller A_{dist} that results in the largest displacement of $\lambda_{\max}$ or maximizes $|\lambda_{\max}(\bar{A}) - \lambda_{\max}(A)|$.

The derivation of $\lambda_{\max}$ w.r.t. perturbation K is given by

$$\frac{\partial \lambda_{\max}}{\partial K} = \frac{v_r^T \frac{\partial \bar{A}}{\partial K} u_r}{v_r^T u_r}, \tag{20}$$

where $r = \arg\max_i \lambda_i$. From A1, $v_r = u_r = [v_{r_1}, \cdots, v_{r_N}]^T$. This results in

$$\frac{\partial \lambda_{\max}}{\partial K} = \frac{v_r^T \frac{\partial \bar{A}}{\partial K} v_r}{\|v_r\|^2}. \tag{21}$$

Assume that the perturbation K works on A_{ij} and A_{ji} of A. Since $\|v_r\|^2 = 1$, we have

$$\frac{\partial \lambda_{\max}}{\partial K} = \begin{bmatrix} v_{r_1} \cdots v_{r_N} \end{bmatrix} \frac{\partial \bar{A}}{\partial K} \begin{bmatrix} v_{r_1} \\ \vdots \\ v_{r_N} \end{bmatrix},$$

with $\left[\frac{\partial \bar{A}}{\partial K}\right]_{ij} = \left[\frac{\partial \bar{A}}{\partial K}\right]_{ji} = \text{sign}(K)$ and are equal to 0 otherwise. After a straightforward calculation, we have

$$\frac{\partial \lambda_{\max}}{\partial K} = 2\text{sign}(K) v_{r_i} v_{r_j}. \tag{22}$$

In order to have $\frac{\partial \lambda_{\max}}{\partial K} < 0$, the sign of K has to be selected as in (19).

We have guaranteed that $\lambda_{\max}$ is always decreasing. Next we find the structure of A_{dist} that minimizes $\lambda_{\max}$ or maximizes $|\lambda_{\max}(\bar{A}) - \lambda_{\max}(A)|$. Thus, the optimization problem (16) can be reformulated as to find the structure of the distributed control law $\bar{A}_{\text{dist}}$ that results in the largest displacement of $\lambda_{\max}$. In other words, we would like to solve the following problem

$$\underset{i,j}{\text{maximize}} \ \left| \frac{\partial \lambda_{\max}}{\partial K} \right|. \tag{23}$$

From (22), the problem (23) can be written as

$$\underset{i,j}{\text{maximize}} \ |v_{r_i} v_{r_j}|.$$

This completes the proof. ■

Proposition 4.2 is easier to solve for a large number of subsystems than (16) since the eigenvector can be computed in a decentralized manner [24], while (16) becomes NP-hard . Moreover it also provides the relation between the communication topology and the eigenvector w.r.t. the rightmost eigenvalue of the interconnected system with the decentralized control law. Note that for $c > 1$, the formulation as in Eq. (18), (19) in Proposition 4.2 will become more complicated. However, by considering an additional assumption given as follows the communication topology design can be simply formulated as follows.

Corollary 4.2: Consider an interconnected systems (13) under assumption A1-A4 with $\bar{A}_i < 0, A_{ij} > 0$ and $K < 0$. The optimal communication topology for a given number c

of links to be added can be reformulated as to find c pairs links between the $i-$th and the $j-$th controller such that the following optimization problem is solved

$$\underset{i,j,\cdots,h,l}{\text{maximize}} \; \overbrace{|v_{r_i}v_{r_j}| + \cdots + |v_{r_h}v_{r_l}|}^{c \text{ pairs}}. \tag{24}$$

Proof: First, we show that under $\bar{A}_i < 0, A_{ij} > 0$ and $K < 0$, the eigenvector corresponding to $\lambda_{\max}$ has all positive or negative entries, i.e., $v_{r_i} > 0$ or $v_{r_i} < 0, \forall i$. From the definition of eigenvector we have

$$Av_r = \lambda_{\max} v_r. \tag{25}$$

The $i-$th row of (25) can be written as

$$\bar{A}_i v_{r_i} + \sum_{j \neq i} A_{ij} v_{r_j} = \lambda_{\max} v_{r_i},$$

where $\bar{A}_i < 0, \lambda_{\max} < 0$ and $A_{ij} > 0$. Rearranging the above equation yields

$$(\bar{A}_i - \lambda_{\max}) v_{r_i} = - \sum_{j \neq i} A_{ij} v_{r_j}. \tag{26}$$

Now let us assume that $v_{r_i} < 0$ and $v_{r_j} > 0, \forall j \neq i$. Thus, in order (26) to be satisfied, the inequality $\bar{A}_i - \lambda_{\max} > 0$ must hold. Assume that $\bar{A}_i > \lambda_{\max}$. It is known that

$$\begin{aligned} \text{trace}(A) &= \sum \lambda_i(A), \\ \lambda_{\max} + \sum_{i \neq r} \lambda_i &= N\bar{A}_i. \end{aligned}$$

Since $\lambda_{\max} > \lambda_i > \cdots > \lambda_N$, we always have

$$\lambda_{\max} + \sum_{i \neq r} \lambda_i < N\bar{A}_i.$$

Thus, (26) will never hold. Therefore, it can be concluded that $\text{sign}(v_{r_i}) = \text{sign}(v_{r_j}), \forall i, j$.

The derivation of $\lambda_{\max}$ is given by

$$\frac{\partial \lambda_{\max}}{\partial K} = \sum \left[\text{sign}(K) v_{r_i} v_{r_j} \right]. \tag{27}$$

Since $\text{sign}(v_{r_i}) = \text{sign}(v_{r_j}), \forall i, j$ and $K < 0$, $\frac{\partial \lambda_{\max}}{\partial K} < 0$. Therefore, the combination of subsystems that result in the largest displacement can be computed by solving (24). ∎

The assumption in Proposition 4.2 is that the feedback gain for the distributed controller is fixed. However, how to choose this gain is not a trivial problem. From $\text{trace}(\bar{A}) = \sum \lambda_i(\bar{A})$ we have $\sum \frac{\partial \lambda_i}{\partial m} = 0$. Thus, when $\frac{\partial \lambda_{\max}}{\partial K} < 0$, there exists at least one eigenvalue of A denoted by $\lambda_m(A)$ such that $\frac{\partial \lambda_m}{\partial K} > 0$. Therefore, for a certain value of K, it is possible that $\lambda_m(\bar{A}) > \lambda_{\max}(A)$, i.e., the system with distributed controller performs worse. Therefore, in order to guarantee the improvement of the performance by using the distributed control, we derive the upper bound for the feedback gain. First we introduce the following Lemma.

Lemma 4.1: Consider an interconnected systems (10) under assumption A1-A4. Let W be a matrix of the right eigenvectors of A in (10). The condition number of W denoted by $\kappa_2(W)$ is one, i.e., $\kappa_2(W) = 1$.

Proof: The condition number is given by

$$\kappa_2(W) = \|W\|_2 \|W^{-1}\|_2. \tag{28}$$

From the definition of eigenvalue and eigenvector we have

$$AW = W \text{diag}(\lambda_i). \tag{29}$$

Taking the transpose and from the assumption $A = A^T$ yields

$$W^T A = \text{diag}(\lambda_i) W^T. \tag{30}$$

From (30) we have $\text{diag}(\lambda_i) = W^T A (W^T)^{-1}$. Substituting this into (29) gives

$$AW = WW^T A (W^T)^{-1}. \tag{31}$$

Equation (31) is satisfied if and only if $WW^T = I$ and $W = (W^T)^{-1}$. Thus we have $W^T = W^{-1}$.

Finally (28) can be computed as

$$\begin{aligned} \kappa_2(W) &= \left[\lambda_{\max}(W^T W)\right]^{\frac{1}{2}} \left[\lambda_{\max}((W^{-1})^T (W^{-1}))\right]^{\frac{1}{2}} \\ &= 1, \end{aligned}$$

which completes the proof. ∎

The upper bound for the distributed feedback gain such that the improvement of the performance for the whole system is guaranteed is given as follows.

Proposition 4.3: Consider an interconnected systems (13) under assumption A1-A4 and $c = 1$. Let $\lambda_m(A)$ be the largest eigenvalue of A that satisfies $\frac{\partial \lambda_m}{\partial K} > 0$. If the feedback gain K satisfies the following inequality,

$$|K| < |\lambda_{\max}(A) - \lambda_m(A)|, \tag{32}$$

then the performance of (13) with distributed controller is guaranteed to be better than the decentralized one.

Proof: From Bauer-Fike theorem [21] and Lemma 4.1 we have

$$\begin{aligned} |\lambda_i - \bar{\lambda}_i| &< \kappa_2(V) \|A_{\text{dist}}\|_2, \\ |\lambda_i - \bar{\lambda}_i| &< \|A_{\text{dist}}\|_2, \end{aligned} \tag{33}$$

where $\bar{\lambda}_i$ are the perturbed eigenvalues. The $\|A_{\text{dist}}\|_2$ can be computed as

$$\|A_{\text{dist}}\|_2 = \left[\lambda_{\max}(A_{\text{dist}}^T A_{\text{dist}})\right]^{\frac{1}{2}} = \left[\lambda_{\max}(A_{\text{dist}} A_{\text{dist}})\right]^{\frac{1}{2}}. \tag{34}$$

Assume that we add the communication links between the $k-$th and $l-$th controllers, i.e., $\bar{A}_{kl} = \bar{A}_{lk} = K$. Since $[A_{\text{dist}} A_{\text{dist}}]_{ij} = \sum_{s=1}^{N} \bar{A}_{is} \bar{A}_{sj}$, $[A_{\text{dist}} A_{\text{dist}}]_{ij} = 0$ except for $[A_{\text{dist}} A_{\text{dist}}]_{kk} = K^2$ and $[A_{\text{dist}} A_{\text{dist}}]_{ll} = K^2$. The eigenvalues of $A_{\text{dist}} A_{\text{dist}}$, i.e., λ are the solutions of $(\lambda - K^2)^2 (\lambda)^{N-2} = 0$. Thus $\lambda \max(A_{\text{dist}} A_{\text{dist}}) = K^2$ and $\|A_{\text{dist}}\|_2 = |K|$. Then (33) becomes

$$|\lambda_i - \bar{\lambda}_i| < |K|.$$

The above inequality tells that the movement of any eigenvalues are bounded by the magnitude of the perturbation to the system. Thus in order to guarantee $\lambda_m(\bar{A}) < \lambda_{\max}(A)$, the feedback gain has to be chosen according to (32). ∎

V. Numerical Example

We consider the following interconnected system consisting of 4 scalar subsystems where it is assumed that the whole systems has been stabilized by the decentralized controllers described in Section III.

$$A = \begin{bmatrix} -15 & 2 & 4 & 3 \\ 2 & -15 & 6 & 0 \\ 4 & 6 & -15 & 3 \\ 3 & 0 & 3 & -15 \end{bmatrix}.$$

The convergence rate of the whole systems with the decentralized controllers, $\mathrm{Re}\{\lambda_{max}(A)\} = -5.5079$. Solving the optimization problem (16) using the YALMIP toolbox [25] for the feedback gain K equal to -1 and $c = 1$, i.e., only one communication link is allowed, gives us the following distributed controller structure.

$$A_{\mathrm{int}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The communication link connects the controller of subsystem 2 and 3 and the convergence rate with the distributed control is improved to $\mathrm{Re}\{\lambda_{max}(\bar{A})\} = -6.1079$. Next we decide where we should add the communication link without solving the optimization problem (16) directly, i.e., by using Proposition 4.2. The eigenvector of $\lambda_{\max}(A)$ is given by $v_r = [0.4813, 0.4992, 0.6293, 0.3510]$. Then the optimization problem (18) is maximized by v_{r_2} and v_{r_3} which gives the value $v_{r_2}v_{r_3} = 0.3141$. Thus the topology that results in a minimum $\lambda_{\max}(A + A_{\mathrm{int}})$ is given by connecting the controller of subsystem 2 and 3 which is similar to the solution of the original optimization problem (16).

VI. Conclusion and Future Works

In this paper we consider the communication topology design of a distributed controller by exploring the additional degree of freedom offered by a communication network in order to improve the performance of a large-scale interconnected system. The strategy is to first design the decentralized controller for each subsystem using the standard methods and improving the performance by allowing the controller of the subsystems to exchange information with each other. In addition to the design of the feedback gain, the communication topology design between the controllers is also considered. The problem is then formulated as a mixed-integer optimization problem. The proposed method guarantees the stability of the overall system under permanent communication link's failure. Furthermore, for a certain class of systems, the optimization problem can be reformulated resulting in a more explicit solution with lower complexity. The ongoing work is focusing on the solution of the optimization problem in a distributed fashion.

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