Sliding Mode Control in Mechanical Systems with Electric Actuators

Sachit Rao, Heide Brandtstädter, Martin Buss, Vadim Utkin

*Abstract-***A control algorithm using the Block Control Principle and Sliding Mode theory is developed for pendulum systems with motor current as control. The behaviour of these non-linear systems with the control as a linear function with saturation is studied in contrast to using the control as a discontinuous function. Results from Singular Perturbation Theory are used towards analysis.**

Index terms- **Affine mechanical systems, Block Control Principle, Motor Current Control, Sliding mode control.**

I. INTRODUCTION

Sliding Mode theory appeared in conjunction with systems being described by differential equations with discontinuities such as variable structure or relay systems. Consider the state-space of an *n*-dimensional affine system with *m*-dimensional vector control [1]:

$$\begin{aligned}&\dot{x} = f(x) + B(x)u \\ &\text{with } x, f \in R^n, u(x) \in R^m \text{ and } B(x) \in R^{n*m}\end{aligned} \tag{1}$$

The control is selected as a discontinuous function of the state; each component u_i, may undergo discontinuities on some surface $s(x)^T = [s_1(x)....s_m(x)]$ such that:

$$u(x) = \begin{cases} u^+(x) \text{ if } s(x) > 0 \\ u^-(x) \text{ if } s(x) < 0 \end{cases} \quad i = 1..m, \tag{2}$$
$$u^T = [u_1(x)....u_m(x)],\ u_i^+(x,t) \neq u_i^-(x,t).$$

Sliding mode may occur in the intersection of *m* surfaces. Using discontinuous control in enforcing sliding modes results in order reduction, decoupling and simplification of the design methodology.

The design procedure should consist of selecting the switching manifold with sliding mode in order to design the desired dynamics of the motion equation and finding a discontinuous control function such that the state reaches the manifold and sliding mode exists in that manifold. The design of control for systems given by (1) can be easily performed for systems in the so-called Regular form. If (1) satisfies certain conditions, such as the existence of a solution to Pfaffian's equation $(\partial\phi / \partial x)B = 0,\ \phi(x) \in R^{n-m}$, then it can be represented in the Regular form [2] given by:

$$\begin{aligned}&\dot{x}_1 = f_1(x_1, x_2, t), \quad x_1 \in R^m \\ &\dot{x}_2 = f_2(x_1, x_2, t) + B_2(x_1, x_2, t)u,\ x_2 \in R^m, \det(B_2) \neq 0\end{aligned} \tag{3}$$

The state sub-vector x_2 is handled as a fictitious control, in the form $x_2 = -s_0(x_1)$, in the first equation of (3) and selected as a function of x_1 to provide the desired dynamics in the first subsystem. In other words, this may be termed as a design problem in the system of the (*n-m*)th order with *m* dimensional control. Then, the control is designed in the form $u = -M(x,t)sign(s)$, component wise, to enforce sliding mode in the manifold

$$s(x_1, x_2) = x_2 + s_0(x_1) = 0 \tag{4}$$

This is the design problem of the *m*th order with *m* dimensional control. After a finite time interval, sliding mode in the manifold (4) starts and the system will exhibit the desired behaviour governed by $\dot{x}_1 = f_1(x_1, -s_0(x_1), t)$. It can be seen that the order of the system is reduced and the motion depends neither on $f_2(x_1, x_2, t)$ nor on the function $B_2(x_1, x_2, t)$ in the second equation of the original system (3).

The design problem has now been decoupled into 2 sub-problems of lower dimensions: selection of fictitious control of the (*n-m*)th order and enforcing sliding mode – of the *m*th order. Note that the second sub-problem looks rather simple since the dimensions of state and control coincide. In the same spirit, decoupling may be continued by partitioning the first equation in (3) into 2 independent problems of (*n*-2*m*)th and *m*th orders which leads to further simplification of the design procedure. This approach developed in [3] is called the Block Control Principle.

At the same time, direct application of the Block Control Principle is not reasonable as it may lead to a large number of blocks. For example, if the system consists of a set of interconnected systems in canonical form, then there is no

Manuscript received April 30, 2004.

Sachit Rao is with the Ohio State University, Department of Mechanical Engineering, Columbus, Ohio, USA. (Email: rao.69@osu.edu).

Heide Brandtstädter is with the the Institute of Automatic Control Engineering (LSR), Technische Universitaet Muenchen, D-80290 Munich, Germany (Email: heide.brandtstaedter@ei.tum.de)

Martin Buss is with the Institute of Automatic Control Engineering (LSR), Technische Universitaet Muenchen, D-80290 Munich, Germany (Email: Martin.Buss@ei.tum.de).

Vadim Utkin is with the Ohio State University, Department of Mechanical Engineering, Columbus, Ohio, USA. (Email: utkin@ee.eng.ohio-state.edu).

need for further decoupling since the design procedure for canonical forms is simple.

The difficulties in the framework of this approach can also be related to unstable zero dynamics. This approach was demonstrated in [4] for mechanical systems under the assumption that control inputs are torques or forces. Bearing in mind that sliding mode implies discontinuous control actions and from an application point of view, this assumption is some kind of idealization. In real systems, electric actuators are commonly used with control as input voltage which can be easily implemented as a discontinuous state function and torque or force as outputs.

In this paper, a design algorithm for control of inverted pendulum systems governed by affine equations is developed using the just described modified Block Control Principle and Sliding Mode Control. Special emphasis will be provided to current control, considering that the pendulum is driven by a DC motor as is offered in [5]. The proposed design method takes into account the limits usually posed on motor current. As an illustration, a simple inverted pendulum will be considered in Section II and simulation results will be provided and as a main case study, in Section III, a rotational inverted pendulum as a plant with unstable zero dynamics will be studied.

II A SIMPLE INVERTED PENDULUM (SIP)

Consider a simple single link inverted pendulum whose equation is given by (5) and the objective is to stabilize the pendulum in the unstable upright position by an input torque τ .

$$J\ddot{\theta} - mgl\sin\theta = \tau \tag{5}$$

As is well known, under the assumption that τ may be selected arbitrarily and $\theta, \dot{\theta}$ are available, the control $\tau = \tau_0 sign(s)$, where $s = c_1\dot{\theta} + c_2\theta$ leads to stabilization at the desired rate of convergence. But this control input – torque, cannot be a discontinuous state function. In a real system, this torque is developed by a DC motor whose equation is:

$$L\frac{di}{dt} + Ri + K_n\dot{\theta} = u,\ \tau = K_m i \tag{6}$$

where i is the motor current, K_n, K_m are DC motor parameters and u is the supply voltage to the motor. If $i = M_i s$ and the control input to the motor is chosen as $u = M_v sign(s_1)$ with $s_1 = i - M_i s$, then for high enough M_i, the system given by (5) with $i = M_i s$ is equivalent to the idealized system with discontinuous control.

In order to prove convergence for the pendulum system and estimate the values of c_1 and c_2, consider a Lyapunov function (7) and its derivative (8) for (5), when $\tau = c_1\dot{\theta} + c_2\theta$.

$$V(\theta) = 0.5\dot{\theta}^2 + k_1\theta^2 + k_2(1+\cos\theta), k_{1,2} > 0 \tag{7}$$

$$\dot{V}(\theta) = c_1\dot{\theta}^2 + \left(c_2 + k_1\right)\theta\dot{\theta} + \left(\left(mgl/J\right) - k_2\right)\sin\theta\dot{\theta} \tag{8}$$

Now, $\theta \to 0$ as $t \to \infty$ if $c_1 < 0, c_2 = -k_1$ and $k_2 = \left(mgl/J\right)$.

If the supply voltage is chosen such that the current achieves the desired value and hence provides the desired torque, then the stabilization problem is solved. This problem assumes that the values of the state variables and their derivatives are available; hence there is no need for an observer. In the more complicated problem in Section III, this assumption is not made and an observer is utilized for control.

If sliding mode is enforced in the surface $s_1 = 0$, the properties of the idealized and real system are close. This is demonstrated by simulation and results are given in figures 2, 3.

III A ROTATIONAL INVERTED PENDULUM SYSTEM

Consider the pendulum driven by a DC motor as given in Figure 1. θ_0 is the angle of the rotational base made with the horizontal, θ_1 is the angle of the pendulum with the vertical and the upright position being zero. The control input to the mechanical system which is torque t and the current i_a cannot be discontinuous. The system equations [4] are given by (9):

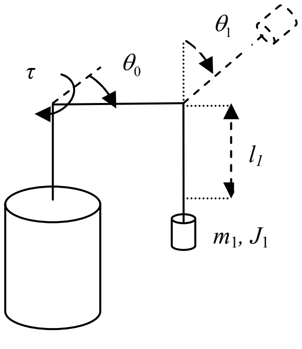

Fig.1 Inverted Pendulum with rotational base (RIP)

$$\begin{aligned}
\ddot{\theta}_0 &= -a_p\dot{\theta}_0 + \frac{K_pK_m}{J_m}i_a \\
\ddot{\theta}_1 &= -\frac{C_1}{J_1}\dot{\theta}_1 + \frac{m_1gl_1}{J_1}\sin\theta_1 - \frac{K_1}{J_1}a_p\dot{\theta}_0 + \tau, \\
\tau &= \frac{K_1K_pK_m}{J_1J_m}i_a,\ L_a\frac{di_a}{dt} = R_ai_a - K_n\dot{\theta}_0 + V_s
\end{aligned} \tag{9}$$

Following the procedure of reducing the system to regular form [2], introduce a new variable: $y = \theta_0 - (J_1/K_1)*\theta_1$ and rewrite the pendulum equation as:

$$\begin{aligned}&\ddot{y} = a_1\dot{\theta}_1 + a_2 \sin\theta_1\\&\ddot{\theta}_1 = b_1\dot{\theta}_1 + b_2\dot{y} + b_3 \sin\theta_1 + \tau;\\&a_1 = (C_1/K_1), a_2 = -(m_1 g l_1/K_1), b_1 = -((C_1/J_1) + a_p),\\&b_2 = -(K_1 a_p/J_1), b_3 = (m_1 g l_1/J_1)\end{aligned} \qquad (10)$$

i_a is the current input to the mechanical system and acts as a fictitious control, whereas the actual control input to the system is the voltage input to the motor V_s. If the fictitious control is selected such that $s = c_1\dot{\theta}_1 + c_2\theta_1 = 0$, then $\theta_1 \to 0$ as $t \to \infty$ [4], but in this case the variable y diverges. This is an illustration of unstable *zero dynamics*. This condition is satisfied by enforcing sliding mode using torque as discontinuous control which is not realistic.

Nevertheless, it is shown that the problem of unstable zero dynamics can be solved by temporarily assuming that torque can be discontinuous. If the control, under this unrealistic assumption, is chosen such that $s_2 = \dot{s}_1 + \alpha_3 s_1 = 0$, where $s_1 = \theta_1 - \alpha_1 y - \alpha_2\dot{y}$, $\alpha_1, \alpha_2, \alpha_3$ are some constants which are to be chosen and $\alpha_3 > 0$, then the variables $\theta_1, y \to 0$ as $t \to \infty$. The proof for this claim is as follows.

With a proper choice of α_3, $s_1, \dot{s}_1 \to 0$ identically and then $\theta_1 = \alpha_1 y + \alpha_2\dot{y}$. Substituting this in the first equation of (10), an independent equation in y is obtained, the behaviour of which needs to be analysed. The first equation of (10) now becomes:

$$\ddot{y} = \frac{a_1\alpha_1}{1 - a_1\alpha_2}\dot{y} + \frac{a_2}{1 - a_1\alpha_2}\sin(\alpha_1 y + \alpha_2\dot{y}), \qquad (11)$$

Choose a Lyapunov function

$$V_y = 0.5\dot{y}^2 + K_y\left(1 + \cos(\alpha_1 y + \alpha_2\dot{y})\right), K_y > 0, \qquad (12)$$

its derivative is of the form

$$\begin{aligned}&\dot{V}_y = \left(a_1\alpha_1/(1 - a_1\alpha_2)\right)\dot{y}^2 + (a_2 - K_y\alpha_1)\dot{y}\sin(\alpha_1 y + \alpha_2\dot{y})\\&-\left(K_y a_2\alpha_2/(1 - a_1\alpha_2)\right)\sin^2(\alpha_1 y + \alpha_2\dot{y})\end{aligned} \qquad (13)$$

In the region $-\pi/2 < \theta_1 < \pi/2, a_1 < 0, a_2 > 0$, by choosing $\alpha_1 = (a_2/K_y) > 0$ and $\alpha_2 > 0$, $y \to 0$ as $t \to \infty$ and hence $\theta_1 \to 0$ as well. Substitution of $\dot{\theta}_1$ as a function of $\ddot{y}$ into the right hand side of the first equation of (10) may give birth to doubts about robustness. It can be shown that the system with an asymptotic observer for evaluation of $\dot{y}$ and $\ddot{y}$ is robust with respect to unmodeled dynamics.

Now, $\dot{s}_2 = f(\theta_1, \dot{\theta}_1, y, \dot{y}) + K^* i_a$, where

$$\begin{aligned}&f = \left[(\alpha_1 + \alpha_2\alpha_3)a_1 + (a_1\alpha_2 - 1)b_1 - \alpha_3\right]\dot{\theta}_1 + [\alpha_1\alpha_3\\&+(a_1\alpha_2 - 1)b_2]\dot{y} + \alpha_2 a_2 \cos\theta_1\dot{\theta} + [(\alpha_1 + \alpha_2\alpha_3)a_2\\&+(a_1\alpha_2 - 1)b_3]\sin\theta_1, K^* = (a_1\alpha_2 - 1)(K_1 K_p K_m/J_1 J_m)\end{aligned} \qquad (14)$$

However, since the torque or current i_a cannot be discontinuous in order to enforce sliding mode on the surface $s_2 = 0$, a linear approximation with saturation in the form $i_a = -M_i s_2$ is used. The stability of this realistic system with continuous current can be shown in the light of Singular Perturbation Theory.

The system can be represented as a slow-fast system in the form:

$$\begin{aligned}&\dot{z} = f_1(z) + B_1 s_2\\&\mu\dot{s}_2 = \mu f_2(z, s_2) - s_2, \mu = 1/M^*, z = [y\ \dot{y}\ \theta_1]^T\end{aligned} \qquad (15)$$

where

$$f_1 = \frac{1}{\alpha_2 a_1 - 1}\left\{\begin{bmatrix}0 & \alpha_2 a_1 - 1 & 0\\ -\alpha_1\alpha_3 a_2 & -(\alpha_1 + \alpha_2\alpha_3)a_2 & \alpha_3 a_2\\ -\alpha_1\alpha_3 & -(\alpha_1 + \alpha_2\alpha_3) & \alpha_3\end{bmatrix} z + \begin{bmatrix}0\\ a_2 - \alpha_2 a_2^2\\ -\alpha_2 a_1\end{bmatrix}\sin\theta_1\right\}, B_1 = \frac{1}{\alpha_2 a_1 - 1}\begin{bmatrix}0 & a_1 & 1\end{bmatrix}^T$$

and (16 i, ii)

$$\begin{aligned}&f_2 = \frac{\left[(\alpha_1 + \alpha_2\alpha_3)a_1 + (a_1\alpha_2 - 1)b_1 - \alpha_3\right]}{(a_1\alpha_2 - 1)}\{[-(\alpha_1 + \alpha_2\alpha_3)\\&((a_1\alpha_2 - 1)b_2 + \alpha_1\alpha_3 - 1)(a_1\alpha_2 - 1)]\dot{y} - \alpha_1\alpha_3 y - \alpha_3\theta_1 +\\&[(\alpha_1 + \alpha_2\alpha_3)a_2 + (a_1\alpha_2 - 1)b_3)(a_1\alpha_2 - 1) - \alpha_2 a_2]\sin\theta_1\} +\\&\frac{a_2\alpha_2\cos\theta_1}{(a_1\alpha_2 - 1)}\{s_2 - (\alpha_1 + \alpha_2\alpha_3)\dot{y} - \alpha_2 a_2 \sin\theta_1 - \alpha_1\alpha_3 y - \alpha_3\theta_1\}\\&M^* = \frac{\left[(\alpha_1 + \alpha_2\alpha_3)a_1 + (a_1\alpha_2 - 1)b_1 - \alpha_3\right]}{(a_1\alpha_2 - 1)} - (a_1\alpha_2 - 1)KM_i\end{aligned}$$

If the constant M^* is very high, the variable s_2 can be taken to be the fast variable and z as the slow one. Since the fast motion is stable, then $s_2 \to 0$ with $\mu \to 0$ and the slow motion is that of the 'ideal' system, which also is stable. The fictitious control i_a can be made to achieve the desired value by enforcing sliding mode in the surface $s = i_a - i_d = 0$, where i_d is the desired current – equal to $-M_i s_2$, with $V_s = M_v sign(s)$.

The analysis until now entailed the use of high gain M^* so that $s_2 = 0$, in what follows, a brief outline to estimate a finite value for M^* will be provided.

Consider the slow variable $z(t)$ in (15), assuming that $s_2 = 0$, and select a Lyapunov function $V(z)_{s_2=0}$ such that $\dot{V}(z)_{s_2=0} = -W_{s_2=0} < 0$. When $\mu = 1/M^*$ has a finite value, then according to Singular Perturbation Theory (SPT), $s_2 = O(\mu)$.

Select another Lyapunov function $V = V(z)_{s_2=0} + 0.5s_2^2$, which will provide an analysis of the overall behaviour of the system given by (15).

Now, $\dot{V} = -W_{s_2=0} + s_2\left(\left(\partial V(z)_{s_2=0} / \partial z\right) + f_2(z, s_2) - M^* s_2\right)$, then $\dot{V} < 0$ if $|s_2| \geq \mu\left|\left(\partial V(z)_{s_2=0} / \partial z\right) + f_2(z, s_2)\right|$ and all the trajectories of (15) are attracted towards a region of order μ. Hence, the error in the trajectories of the system with high gain M^* and a finite value of M^* will be of order μ. In this region, the system can be linearised and its behaviour can be studied. Again as it follows from SPT, these linear systems will have eigenvalues with negative real parts and may be partitioned into 'slow' and 'fast' ones which again imply system stability.

The simulation results for this system with an asymptotic full order observer are in figures 4, 5 and 6.

IV SIMULATION RESULTS

The simulation for the two pendulum systems considered here was carried out MATLAB/Simulink with the following parameters – solver: Runge-Kutta and sampling time: 0.0001 sec for the SIP and 0.001 sec for the RIP.

The SIP has the following system parameters: length of the pendulum = 0.1*m*, mass of the pendulum = 0.01 *kg*, moment of inertia of the pendulum = 0.01 *N-m-s*^{2}, motor torque constant = 0.0302 *N-m/A*, back emf constant = 60/317 *Vs,* motor inertia = 1.34e-6 *N-m-s*^{2}. The motor parameters for this system are: R = 0.316 W, L = 0.0008 *H*. Figure 2 shows the variation of the pendulum angle θ and Figure 3 shows the motor current *i* and supply voltage *u*. The current *i* is chosen in the form M_i sat(s) with M_i = 80 and a saturation of ±0.2 , the control u = M_v sat(s_1) with M_v = -25, a saturation of ±0.1 and c_1 = -0.65, c_2 = -4.5.

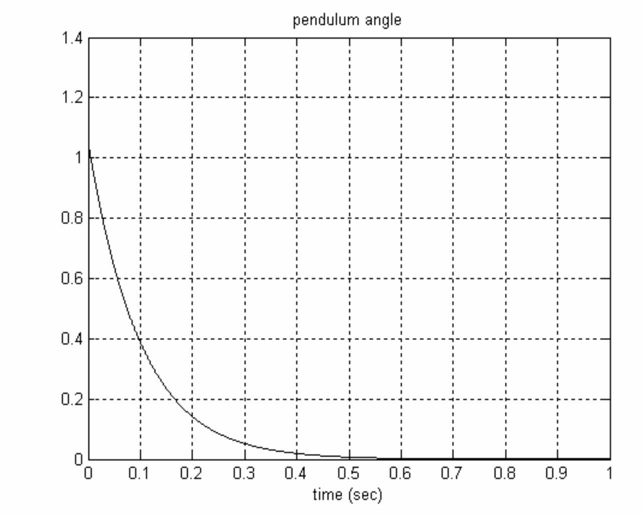

Fig. 2 The SIP angle

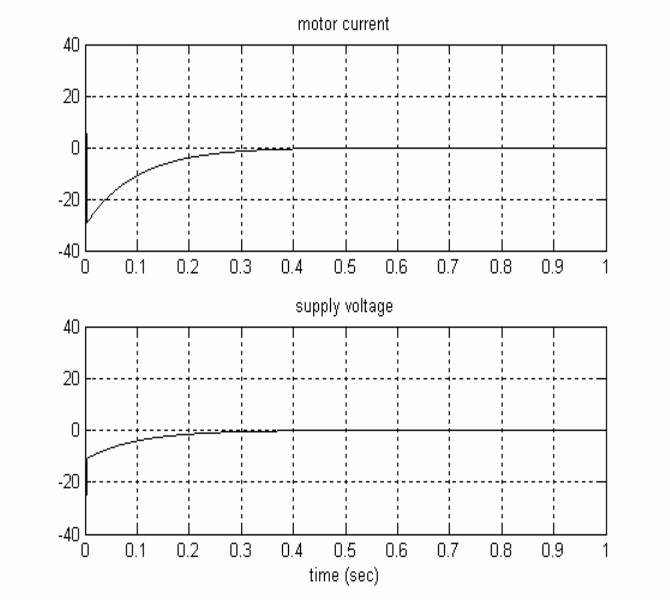

Fig.3 DC Motor Current and Supply Voltage for the SIP

The RIP considered in Section III has the following parameters: l_1 – length of the pendulum = 0.113 *m*, m_1 – mass of the pendulum = 0.0862 *kg*, J_1 – moment of inertia of the pendulum = 0.0013 *N-m-s*^{2}, a_p = motor constant = 33.04, C_1 – frictional constant = 0.00297 *(N-m-s)/rad*, K_m = motor torque constant = 0.3 *N-m/A*, K_n = back emf constant = 0.1 *Vs*, J_m = moment of inertia of the motor = 3*J_1 and the proportionality constant K_1 is given by: $\begin{cases} K_1 = -0.0019, \text{ if } -\pi/2 < \theta_1 < \pi/2 \\ K_1 = +0.0019, \text{ elsewhere.} \end{cases}$.

Figure 4 shows the variation of the pendulum and base angles, a comparison is made with the observed and actual state variables. It is seen that there is fast convergence of the estimated with the actual values.

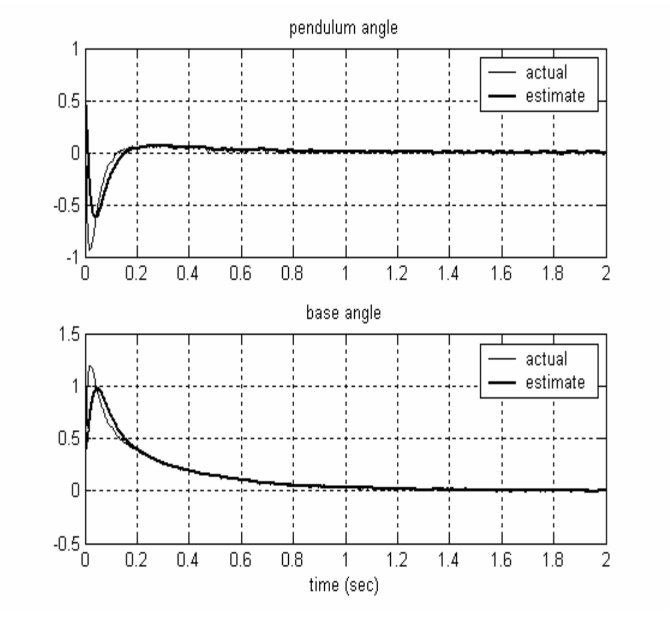

Fig. 4 The Inverted Pendulum and Base angle of the RIP

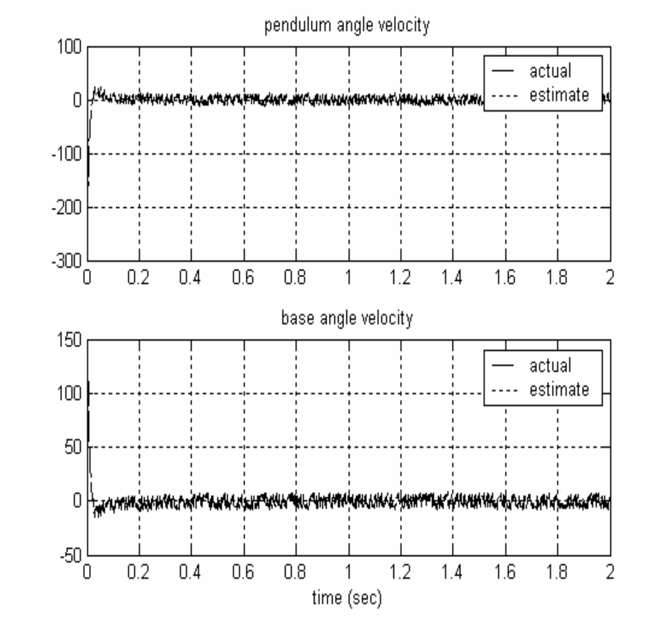

Fig. 6 Pendulum and Base angle velocities for the RIP

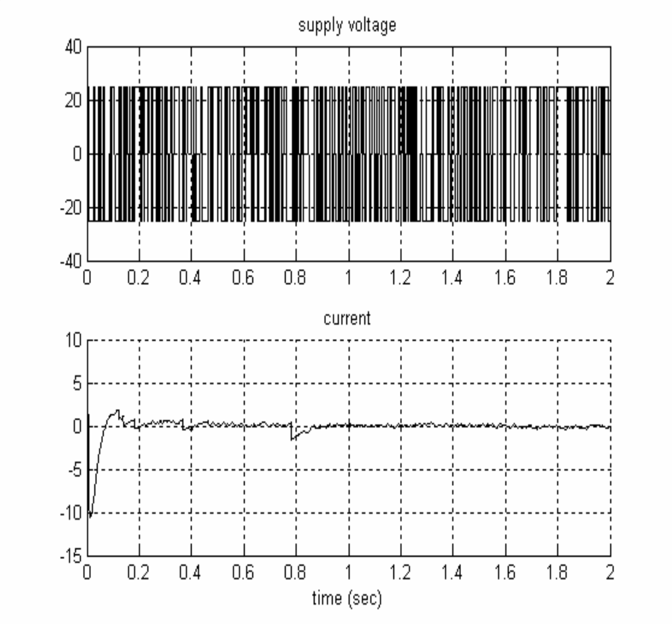

Fig. 6 Supply Voltage and DC Motor Current for the RIP

The current i_a is chosen in the form $M_i\,\mathrm{sat}(s_2)$ with $M_i = 10$ and a saturation of ± 10, the control $V_s = M_v\,\mathrm{sign}(s)$ with $M_v = -25$ and $\alpha_1 = 0.9, \alpha_2 = 0.35, \alpha_3 = 100$. The observer used is a full order observer with the states being $\hat{y}, \hat{\dot{y}}, \hat{\theta}_1, \hat{\dot{\theta}}_1$ and the observer gain matrix is $L = [75\ 15,\ 25\ 20,\ 35\ 60,\ 60\ 30]^T$.

Arbitrary initial conditions are chosen for both the cases studied.

V CONCLUSIONS

The use of the modified Block Control Principle with Sliding Mode theory is demonstrated for mechanical systems in order to tackle the problem of unstable zero dynamics and use of electric actuators for control. Some results based on Singular Perturbation Theory have proved to be useful in developing algorithms in such cases where the control input cannot be discontinuous and to estimate the error in the trajectories between the ideal and realistic systems. Simulation results have shown that the problem can be solved even in cases where all state variables are not available.

VI REFERENCES

[1] V.Utkin, J.Guldner and J.Shi, *Sliding Mode Control in Electro-Mechanical Systems*, Taylor & Francis, 1999.

[2] A.Luk'yanov and V.Utkin, "Methods of Reducing Equations for Dynamic Systems to a Regular Form", *Automation and Remote Control*, 42, pp.413-420, 1981.

[3] S.Drakunov, D.Izosimov, A.Luk'yanov, V.A.Utkin, V.I.Utkin, "Block Control Principle I", *Automation and Remote Control*, 51, pp. 601-609, 1990.

[4] V.Utkin, D.Chen, H.Chang, "Block Control Principle for Mechanical Systems", *JDSMC*, vol.122, No.1, pp.1-10, 2000.

[5] A. Sabanovic, "Variable Structure Controllers in Motion-Control Systems", *Control Engineering Practice*, Vol.1, No.3, pp. 519-528, 1993.