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Active mode observation of switching systems based on set-valued estimation of the continuous state

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SUMMARY

Mode observability is addressed for a class of discrete-time linear systems that may switch in an unknown and unpredictable way among different modes taken from a finite set. The possible *a priori* knowledge on the continuous state of the system and the presence of unknown but bounded noises affecting both the system and the measurement equations are explicitly taken into account. The mode observation is performed 'actively': control sequences (discerning control sequences) are searched, which allow to identify the switching sequence on the basis of the observations. Conditions that characterize discerning controls in a finite-horizon setting are obtained. Moreover, a procedure is proposed in order to derive 'persistently discerning' control sequences (over an infinite horizon). A numerical example is reported to clarify the approach. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Hybrid systems have recently received much attention from the control community not only for an academic interest but also for the practical importance they hold, as they can be used to describe a wide range of physical and engineering systems. The diffusion of the paradigm of hybrid systems has brought to the attention of the scientific community the necessity of a methodological study of the structural properties (e.g. reachability, stabilizability, observability, and detectability) of such

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systems. With this respect, many notable complications arise since for different classes of hybrid systems, specific definitions of such structural properties can be given. If we restrict our attention on the observability, for jump linear systems, i.e. systems in which the evolution of the discrete dynamics is governed by the Markov processes, the reader is referred to [1] for a survey and to [2, 3] for recent developments. For piece-wise affine systems, where the discrete state is a piece-wise function of the previous continuous state, the reader is referred to [4, 5]. Finally, for systems where the control input is extended to the discrete state (the discrete state is a control variable) the reader is referred to [6].

In this paper, we focus on a particular class of hybrid systems: *switching discrete-time linear systems*, i.e. linear systems in which the matrices governing the system and the measurement equations can take values in a finite set (the index denoting such values being called the 'mode' or the 'discrete state'). We assume that the system matrices may switch at each time instant in an unknown and unpredictable way.

It should be evident that, for such a class of systems, the possibility of reconstructing the past discrete states on the basis of the available observations may be of interest in many practical applications (e.g. fault detection, estimation of the continuous state, and control). With this respect, in this paper, we investigate the connection between the choice of the control sequence and the observability of the system mode. More specifically, the problem we address consists in looking for suitable control sequences (*discerning control sequences*) such that the switching sequence can be reconstructed on the basis of the observations. We call this problem *active mode observation*, borrowing the term 'active' from the literature on active estimation/identification (see [7–9]). It is worth noting that, in a probabilistic framework, a similar problem has been recently addressed in [10] for a different class of hybrid systems, for which the evolution of the discrete state is supposed to be governed by a Markov chain and the noise vectors are normally distributed random variables.

A first systematic attempt to the study of mode observability for unforced noise-free switching linear systems was made in [11]. In [12], such results were extended to comply with the presence of bounded disturbances that corrupt the dynamics and the measures. In [13, 14], the effects of the choice of the control sequence on the observability of the switching sequence were investigated in a noise-free setting. More recent advances on this topic have been developed in [15], where a characterization of discerning control sequence in the presence of unknown but bounded noises was derived. Note that in [15], an extension to the infinite-horizon setting was also considered.

The results of [15] were derived assuming that no *a priori* information on the initial continuous state is available. However, in many practical cases, the *a priori* knowledge of the system may lead one to consider a restricted set of 'admissible' initial states. For example, such information may be derived on the basis of physical considerations on the real system. In this paper, the results of [15] are extended in order to derive conditions for active mode observability, which take into account such possible *a priori* information.

First, in Section 2, the active mode-observation problem is addressed over a finite horizon. Notice that this case is of interest whenever an ad hoc manoeuvre can be implemented on a plant in order to identify the actual operating mode (e.g. for fault detection purposes). Necessary and sufficient conditions for the existence of discerning control sequences are derived, which can be expressed in terms of an easy-to-check 'rank-based' test. Such novel conditions turn out to be much easier to satisfy than those of [13, 15], especially in the quite common case wherein the initial continuous state is known to belong to a bounded set. In particular, it can be shown that the knowledge of a set to which the initial continuous state belongs can be crucial to obtain discerning control sequences and, more generally, it allows for a larger set of discerning control sequences.

On the basis of such results, a receding-horizon procedure is developed in Section 3, which allows to derive a persistently discerning control sequence when an infinite-horizon active mode-observation problem is considered. Such a procedure relies on the combination of a recursive set-valued estimation algorithm (that propagates the information on the continuous state over time) with a moving-horizon scheme for the observation of the switching sequence. In Section 4, a numerical example is presented to show how the proposed methodology can be applied. For the reader's convenience the proofs are reported in the Appendix.

2. MODE OBSERVABILITY OVER A FINITE HORIZON

Let us consider switching discrete-time linear systems described by

$$x_{t+1} = A(\lambda_t)x_t + B(\lambda_t)u_t + w_t \tag{1a}$$

$$y_t = C(\lambda_t)x_t + v_t \tag{1b}$$

where $t=0,1,\ldots$ is the time instant, $x_t \in \mathbb{R}^n$ is the continuous-state vector (the initial state x_0 is unknown), $\lambda_t \in \mathscr{L} \triangleq \{1,2,\ldots,L\}$ is the discrete state (or mode of the system), $u_t \in \mathbb{R}^k$ is the control vector, $w_t \in \mathscr{W} \subset \mathbb{R}^n$ is the system noise vector, $y_t \in \mathbb{R}^m$ is the vector of the measurements, and $v_t \in \mathscr{V} \subset \mathbb{R}^m$ is the measurement noise vector. $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$, $\lambda \in \mathscr{L}$, are $n \times n$, $n \times k$, and $m \times n$ matrices, respectively.

The initial state x_0 and the noise vectors w_t and v_t are supposed to take their values in the known sets $\mathscr{X}_0$, $\mathscr{W}$, and $\mathscr{V}$, respectively. We assume that no other *a priori* information is available on such quantities (i.e. their statistics are supposed to be unknown) and that the law governing the evolution of the discrete state is unknown.

In order to address the active mode observability problem over a finite horizon $[0,N]$, some preliminary definitions are needed. Given a generic sequence $\mathbf{z}_{0,\infty} \triangleq \{z_t; t=0,1,\ldots\}$ and two time instants $t_1 \leqslant t_2$, we define $\mathbf{z}_{t_1,t_2} \triangleq \mathrm{col}(z_{t_1}, z_{t_1+1}, \ldots, z_{t_2})$.

By defining the quantities

$$F(\boldsymbol{\lambda}_{0,N}) \triangleq \begin{bmatrix} C(\lambda_0) \\ C(\lambda_1)A(\lambda_0) \\ \vdots \\ C(\lambda_N)\prod_{i=1}^{N} A(\lambda_{N-i}) \end{bmatrix}$$

$$H(\boldsymbol{\lambda}_{0,N}) \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C(\lambda_1) & 0 & \cdots & 0 \\ C(\lambda_2)A(\lambda_1) & C(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C(\lambda_N)\prod_{i=1}^{N-1} A(\lambda_{N-i}) & C(\lambda_N)\prod_{i=1}^{N-2} A(\lambda_{N-i}) & \cdots & C(\lambda_N) \end{bmatrix}$$

$$G(\boldsymbol{\lambda}_{0,N}) \triangleq H(\boldsymbol{\lambda}_{0,N})\,\mathrm{diag}[B(\lambda_0),\ldots,B(\lambda_{N-1})]$$

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the observation sequence $\mathbf{y}_{0,N}$ in the interval $[0, N]$ can be written as

$$\mathbf{y}_{0,N} = F(\boldsymbol{\lambda}_{0,N})x_0 + G(\boldsymbol{\lambda}_{0,N})\mathbf{u}_{0,N-1} + H(\boldsymbol{\lambda}_{0,N})\mathbf{w}_{0,N-1} + \mathbf{v}_{0,N}$$

Furthermore, let us denote by $\mathscr{Y}(\boldsymbol{\lambda}_{0,N}, \mathbf{u}_{0,N-1}, \mathscr{X}_0)$ the set of all the possible observation sequences associated with the switching sequence $\boldsymbol{\lambda}_{0,N}$ and the control sequence $\mathbf{u}_{0,N-1}$ for any possible initial continuous state $x_0 \in \mathscr{X}_0$ and any possible noise sequence, i.e.

$$\mathscr{Y}(\boldsymbol{\lambda}_{0,N}, \mathbf{u}_{0,N-1}, \mathscr{X}_0) \triangleq \{\mathbf{y} \in \mathbb{R}^{m(N+1)} : \mathbf{y} = F(\boldsymbol{\lambda}_{0,N})x + G(\boldsymbol{\lambda}_{0,N})\mathbf{u}_{0,N-1} + H(\boldsymbol{\lambda}_{0,N})\mathbf{w} + \mathbf{v},$$
$$x \in \mathscr{X}_0, \mathbf{w} \in \mathscr{W}^N, \mathbf{v} \in \mathscr{V}^{N+1}\}$$

In this section, we want to investigate whether it is possible to choose the control sequence $\mathbf{u}_{0,N-1}$ in such a way that the switching sequence $\boldsymbol{\lambda}_{0,N}$ can be reconstructed on the basis of the observation sequence $\mathbf{y}_{0,N}$ and of the control sequence $\mathbf{u}_{0,N-1}$ for any possible initial continuous state and any possible noise sequence.

If the evolution of the discrete state is completely unpredictable, the switching sequence $\boldsymbol{\lambda}_{0,N}$ may assume any value in the set $\mathscr{L}^{N+1}$. However, in many practical cases, the *a priori* knowledge of the system may allow one to consider a restricted set of 'admissible' switching sequences. For example, there might be a minimum dwell time (i.e. a minimum number of time instants between consecutive switches) or the evolution of the discrete state might be governed by a hidden finite state machine. Of course, such *a priori* knowledge may make the task of identifying the discrete state from the measurements $\mathbf{y}_{0,N}$ considerably simpler. As a consequence, instead of considering all the possible switching sequences belonging to $\mathscr{L}^{N+1}$, one should consider a restricted set $\mathscr{P}_N \subseteq \mathscr{L}^{N+1}$ consisting of all the switching sequences compatible with the *a priori* knowledge of the evolution of the discrete state. This would add no theoretical difficulty but some notational complication. Hence, to simplify the presentation, in the following we shall always suppose the law governing the evolution of the discrete state completely unknown.

Unfortunately, as shown in [12] for autonomous systems, in general it is impossible to uniquely determine the mode of the system at the beginning and at the end of an observation window (see Example 1 below). Then, we introduce two integers, α and ω, with $\alpha, \omega \geqslant 0$ and $\alpha + \omega \leqslant N$, such that we can try to uniquely determine the discrete state λ_t in the restricted interval $[\alpha, N-\omega]$ on the basis of the observation sequence $\mathbf{y}_{0,N}$ and of the control sequence $\mathbf{u}_{0,N-1}$. Given a switching sequence $\boldsymbol{\lambda}$ in the interval $[0, N]$, let us denote by $r^{\alpha,\omega}(\boldsymbol{\lambda})$ the restriction of $\boldsymbol{\lambda}$ to the interval $[\alpha, N-\omega]$.

Summing up, the following problem shall be addressed.

Problem 1 (Active Mode Observation)
Given a time horizon N, determine, if it exists, a control sequence $\mathbf{u}_{0,N-1}$ such that, for some $\alpha, \omega > 0$ with $\alpha + \omega < N$, we have

$$\mathscr{Y}(\boldsymbol{\lambda}, \mathbf{u}_{0,N-1}, \mathscr{X}_0) \cap \mathscr{Y}(\boldsymbol{\lambda}', \mathbf{u}_{0,N-1}, \mathscr{X}_0) = \emptyset \tag{2}$$

for every pair of switching sequences $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$.

With respect to Problem 1 the following definitions can be given:

Definition 1

A control sequence $\mathbf{u}_{0,N-1}$ that solves Problem 1 for some α and ω is called an (N,α,ω)*-discerning control sequence* for system (1).

If there exists an (N,α,ω)-discerning control sequence (i.e. if Problem 1 is feasible for some N, α, and ω), then system (1) is said to be (N,α,ω)*-mode observable*.

In other words, if $\mathbf{u}_{0,N-1}$ is an (N,α,ω)-discerning control sequence then different switching sequences in the interval $[\alpha, N-\omega]$ generate different observation sequences in the interval $[0,N]$, regardless of the initial states and the noise sequences. Thus, the switching sequence $\boldsymbol{\lambda}_{\alpha,N-\omega} = r^{\alpha,\omega}(\boldsymbol{\lambda}_{0,N})$ can be uniquely determined from the observation sequence $\mathbf{y}_{0,N}$.

Although it would be preferable to have $\alpha=\omega=0$, the following counterexample shows that this is not always possible:

Example 1

Suppose $0\in\mathscr{X}_0$. In this case, the set

$$\mathscr{Y}(\boldsymbol{\lambda},\mathbf{u}_{0,N-1},0)=\{\mathbf{y}\in\mathbb{R}^{m(N+1)}:\mathbf{y}=G(\boldsymbol{\lambda})\mathbf{u}_{0,N-1}+H(\boldsymbol{\lambda})\mathbf{w}+\mathbf{v},\mathbf{w}\in\mathscr{W}^N,\mathbf{v}\in\mathscr{V}^{N+1}\}$$

belongs to the set $\mathscr{Y}(\boldsymbol{\lambda},\mathbf{u}_{0,N-1},\mathscr{X}_0)$ of possible observation sequences associated with each $\boldsymbol{\lambda}\in\mathscr{L}^{N+1}$.

First, consider two feasible switching sequences $\boldsymbol{\lambda}'_{0,N},\boldsymbol{\lambda}''_{0,N}$ that differ only in the first mode, i.e. $\boldsymbol{\lambda}'_{1,N}=\boldsymbol{\lambda}''_{1,N}$ and $\lambda'_0\neq\lambda''_0$. If $B(\lambda'_0)=B(\lambda''_0)$, one has $H(\boldsymbol{\lambda}'_{0,N})=H(\boldsymbol{\lambda}''_{0,N})$ and $G(\boldsymbol{\lambda}'_{0,N})=G(\boldsymbol{\lambda}''_{0,N})$. Then, in this particular case, $\mathscr{Y}(\boldsymbol{\lambda}'_{0,N},\mathbf{u}_{0,N-1},0)=\mathscr{Y}(\boldsymbol{\lambda}''_{0,N},\mathbf{u}_{0,N-1},0)$, which implies that the set of possible outputs $\mathscr{Y}(\boldsymbol{\lambda}'_{0,N},\mathbf{u}_{0,N-1},\mathscr{X}_0)$ and $\mathscr{Y}(\boldsymbol{\lambda}''_{0,N},\mathbf{u}_{0,N-1},\mathscr{X}_0)$ associated with $\boldsymbol{\lambda}'_{0,N}$ and $\boldsymbol{\lambda}''_{0,N}$ (respectively) cannot be made disjoint. Thus, the first mode cannot be uniquely determined and, in order to have (N,α,ω)-mode observability, the constant α has to be greater than one.

Similarly, suppose now that $\boldsymbol{\lambda}'_{0,N}$ and $\boldsymbol{\lambda}''_{0,N}$ differ only in the last mode, i.e. $\boldsymbol{\lambda}'_{0,N-1}=\boldsymbol{\lambda}''_{0,N-1}$ and $\lambda'_N\neq\lambda''_N$. If $C(\lambda'_N)=C(\lambda''_N)$, again, one has $H(\boldsymbol{\lambda}'_{0,N})=H(\boldsymbol{\lambda}''_{0,N})$ and $G(\boldsymbol{\lambda}'_{0,N})=G(\boldsymbol{\lambda}''_{0,N})$, which implies $\mathscr{Y}(\boldsymbol{\lambda}'_{0,N},\mathbf{u}_{0,N-1},0)=\mathscr{Y}(\boldsymbol{\lambda}''_{0,N},\mathbf{u}_{0,N-1},0)$ and $\mathscr{Y}(\boldsymbol{\lambda}'_{0,N},\mathbf{u}_{0,N-1},\mathscr{X}_0)\cap\mathscr{Y}(\boldsymbol{\lambda}''_{0,N},\mathbf{u}_{0,N-1},\mathscr{X}_0)\neq\emptyset$ for any possible control sequence $\mathbf{u}_{0,N-1}$. In this case, the last mode cannot be uniquely determined and, in order to have (N,α,ω)-mode observability, the constant ω has to be greater than one.

It is important to point out that, in general, the knowledge about the initial continuous state can be crucial for the existence of feasible solutions to Problem 1. These considerations are illustrated by means of the following elementary example:

Example 2

Consider a simple scalar linear switching system described by Equations (1) with

$$\begin{aligned} A(1)&=a', \quad B(1)=b', \quad C(1)=c' \\ A(2)&=a'', \quad B(2)=b'', \quad C(2)=c'' \end{aligned} \tag{3}$$

For the sake of simplicity, suppose that $N=1$ and let the set of admissible switching sequences be $\{(1,1),(2,2)\}$. Moreover, it is assumed that the measurement noise v_t is identically null, and that the system noise w_t is unknown but bounded, i.e. $w_t\in\mathscr{W}=[-\rho_w,\rho_w]$ with $\rho_w>0$.

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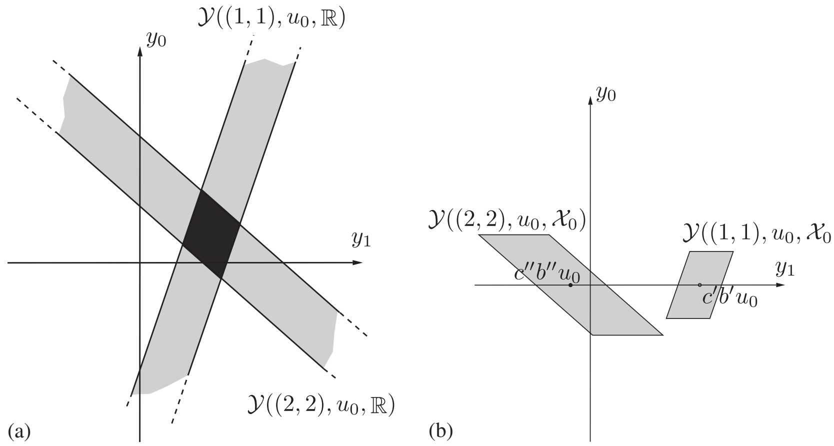

Figure 1. Plots of the sets $\mathscr{Y}(\boldsymbol{\lambda}_{0,N}, \mathbf{u}_{0,N-1}, \mathscr{X}_0)$ considered in Example 1.

Suppose first that no information is available about the initial continuous state, i.e. $\mathscr{X}_0 = \mathbb{R}$. In this case, the two sets $\mathscr{Y}((1,1), u_0, \mathbb{R})$ and $\mathscr{Y}((2,2), u_0, \mathbb{R})$ correspond to two stripes in the plane (y_0, y_1) (see Figure 1(a)): the stripes' directions are determined by the two observability matrices $F(1,1) = [c', c'a']^\top$ and $F(2,2) = [c'', c''a'']^\top$ whereas the control u_0 only influences the stripes' positions. As a consequence, unless the two stripes are parallel (i.e. unless $a' = a''$), it is not possible to choose u_0 so that the two stripes are completely disjoint, i.e. Problem 1 has no feasible solutions.

Suppose now that the initial continuous state is known to belong to a bounded interval, i.e. $\mathscr{X}_0 = [-\rho_x, \rho_x]$ with $\rho_x > 0$. In this case, the two sets $\mathscr{Y}((1,1), u_0, \mathscr{X}_0)$ and $\mathscr{Y}((2,2), u_0, \mathscr{X}_0)$ take the form of two parallelograms centred in $c'b'u_0$ and $c''b''u_0$, respectively (see Figure 1(b)). Thus, provided that $c'b' \neq c''b''$, it is possible to make such sets completely disjoint by choosing a suitably large control u_0.

Clearly, in order to derive meaningful (and computationally tractable) conditions some restriction on the form of the set $\mathscr{X}_0$, to which the initial state x_0 belongs, has to be introduced. Both in the contexts of model predictive control and set membership estimation (see, e.g. [16, 17]) it is common to assume that the initial continuous state belongs to a known bounded set. Usually, in these cases the set $\mathscr{X}_0$ is supposed to have a very specific structure (e.g. polytopic or ellipsoidal). To account for more general situations, the following assumption can be stated. (Given two sets $\mathscr{A}$ and $\mathscr{A}'$, let us denote the Minkowski sum of the two sets by $\mathscr{A} \oplus \mathscr{A}' \triangleq \{v = a + a'; a \in \mathscr{A}, a' \in \mathscr{A}'\}$).

Assumption A1
The set $\mathscr{X}_0$ can be decomposed as the Minkowski sum of a linear subspace $\mathscr{S}$ and a compact set $\mathscr{B}$, i.e.

$$\mathscr{X}_0 = \mathscr{S} \oplus \mathscr{B} \tag{4}$$

Note that an equality constraint of the form $\mathscr{X}_0 = \{x_0 : Ex_0 = f\}$ can also be written as in (4). In fact, provided that a solution to the system of linear equations $Ex_0 = f$ exists (i.e. $\mathscr{X}_0 \neq \emptyset$), then the set $\mathscr{X}_0$ of all the solutions can be obtained as the Minkowski sum of the linear subspace

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$\ker(E)$ with the vector $E^\top(EE^\top)^{-1}f$ (here, without loss of generality, the matrix E is supposed to have full rank). The interested reader is referred to [18].

Of course, even if Assumption A1 is rather general, there are sets of initial conditions that cannot be described as in (4). Examples are non-trivial conic sets (e.g. $\mathscr{X}_0$ be the positive orthant $\mathbb{R}^n_+$). These sets of initial conditions are of interest, for example, in the theory of positive systems.

We would also like to point out that all the results that will be given hereafter could be easily extended to the more general framework in which the set $\mathscr{X}_0$ can be decomposed as the union of a finite number of sets of the form $\mathscr{S}_i \oplus \mathscr{B}_i$, i.e.

$$\mathscr{X}_0 = \bigcup_{i=1}^{K} (\mathscr{S}_i \oplus \mathscr{B}_i)$$

where the sets $\mathscr{S}_i, i=1,\ldots,K$ are linear subspaces and the sets $\mathscr{B}_i, i=1,\ldots,K$ are generic bounded sets.

As to the sets $\mathscr{W}$ and $\mathscr{V}$, the following assumption is made:

Assumption A2
The sets $\mathscr{W}$ and $\mathscr{V}$ are compact.

The next theorem provides a characterization of all the discerning control sequences in the considered framework.

Theorem 1
Suppose that Assumptions A1 and A2 are satisfied. Then a control sequence $\mathbf{u}_{0,N-1}$ is (N,α,ω)-discerning if and only if, for every $\lambda, \lambda' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda) \neq r^{\alpha,\omega}(\lambda')$, we have

$$[I-P(\lambda,\lambda')][G(\lambda)-G(\lambda')]\mathbf{u}_{0,N-1} \notin \mathscr{Q}(\lambda,\lambda',\mathscr{X}_0) \tag{5}$$

where $P(\lambda,\lambda')$ is the matrix of the orthogonal projection on the linear subspace $\{F(\lambda)\mathscr{S}\} \oplus \{F(\lambda')\mathscr{S}\}$ and each $\mathscr{Q}(\lambda,\lambda',\mathscr{X}_0)$ is a bounded set obtained as

$$\begin{aligned}\mathscr{Q}(\lambda,\lambda',\mathscr{X}_0) = [I-P(\lambda,\lambda')]\{&[F(\lambda')\mathscr{B}] \oplus [-F(\lambda)\mathscr{B}]\\ &\oplus [H(\lambda')\mathscr{W}^N] \oplus [-H(\lambda)\mathscr{W}^N] \oplus \mathscr{V}^{N+1} \oplus [-\mathscr{V}^{N+1}]\}\end{aligned} \tag{6}$$

In the special case when $\mathscr{S}=\{0\}$, the projection matrix $P(\lambda,\lambda')$ results to be null. Then condition (5) reduces to

$$[G(\lambda)-G(\lambda')]\mathbf{u}_{0,N-1} \notin \mathscr{Q}(\lambda,\lambda',\mathscr{X}_0) \tag{7}$$

In order to shed some light on the possibility of finding a control sequence that satisfies condition (5), the following corollary to Theorem 1 can be stated:

Corollary 1
Suppose that Assumptions A1 and A2 are satisfied. Moreover, suppose that there exists a control sequence $\mathbf{u}'_{0,N-1}$ such that, for every $\lambda, \lambda' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda) \neq r^{\alpha,\omega}(\lambda')$ and $0 \in \mathscr{Q}(\lambda,\lambda',\mathscr{X}_0)$,

$$[I-P(\lambda,\lambda')][G(\lambda)-G(\lambda')]\mathbf{u}'_{0,N-1} \neq 0 \tag{8}$$

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Then the control sequence $\mathbf{u}_{0,N-1} = g\mathbf{u}'_{0,N-1}$ satisfies condition (5) and, therefore, is an (N, α, ω)-discerning control sequence for system (1) for any $g>0$ with the exception of a set with bounded measure (given by the union of a finite number of intervals. See the proof).

Corollary 1 not only provides a sufficient condition for the (N, α, ω)-mode observability of system (1), but it also suggests that an (N, α, ω)-discerning control sequence can be obtained by means of the following simple procedure: (i) determine a control sequence $\mathbf{u}'_{0,N-1}$ satisfying condition (8); (ii) scale the control sequence $\mathbf{u}'_{0,N-1}$ by means of a suitable scalar g (see the proof).

Such a result is instrumental in deriving a simple rank condition on the matrix $G(\boldsymbol{\lambda}) - G(\boldsymbol{\lambda}')$, which ensures the (N, α, ω)-mode observability of system (1). More specifically, the following theorem can be stated:

Theorem 2

Suppose that Assumptions A1 and A2 are satisfied. Moreover, let S be a basis matrix of the linear subspace $\mathscr{S}$. Then, system (1) is (N, α, ω)-mode observable if and only if for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$ and $0 \in \mathscr{Z}(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0)$

$$\mathrm{rank}[F(\boldsymbol{\lambda})S | F(\boldsymbol{\lambda}')S | G(\boldsymbol{\lambda}) - G(\boldsymbol{\lambda}')] > \mathrm{rank}[F(\boldsymbol{\lambda})S | F(\boldsymbol{\lambda}')S] \tag{9}$$

A few remarks about Theorem 2 are now in place.

Remark 1

It should be evident that, as expected, the smaller the dimension of the linear space $\mathscr{S}$ the easier is to satisfy condition (9). For instance, when $\mathscr{S} = \{0\}$ (i.e. the initial continuous state is known to belong to a bounded set), condition (9) reduces to

$$\mathrm{rank}[G(\boldsymbol{\lambda}) - G(\boldsymbol{\lambda}')] > 0 \tag{10}$$

Thus, in this special case (which is quite common in practice) it is possible to discern between two different switching sequences $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ provided that they give rise to different *N-step behavior matrices* $G(\boldsymbol{\lambda})$ and $G(\boldsymbol{\lambda}')$.

Remark 2

When $0 \in \mathscr{B}$ and $0 \in \mathscr{W}$, then $0 \in \mathscr{Z}(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0)$ for every pair $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathscr{L}^{N+1}$ regardless of the form of the set $\mathscr{X}_0$. As a consequence, in this case, system (1) is (N, α, ω)-mode observable if and only if condition (9) is satisfied for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$.

As a final remark, it is worth noting that, in general, determining exactly the set of discerning control sequences might be a difficult task. However, in the special case in which the sets $\mathscr{B}$, $\mathscr{W}$, and $\mathscr{V}$ are polytopes, such a task becomes quite simple. With this respect, first note that in this case each set $\mathscr{Z}(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0)$ is also a polytope. Therefore, it is possible to find a suitable matrix $\Psi(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0)$ and a suitable vector $\mu(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0)$ such that

$$\mathscr{Z}(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0) = \{\mathbf{z} \in \mathbb{R}^{m(N+1)} : \Psi(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0)\mathbf{z} \leqslant \mu(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0)\}$$

This, in turn, implies that for any pair $\boldsymbol{\lambda}, \boldsymbol{\lambda}'$ the set $\mathscr{P}(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0)$ of all the control sequences $\mathbf{u}_{0,N-1}$ satisfying condition

$$[I - P(\boldsymbol{\lambda}, \boldsymbol{\lambda}')][G(\boldsymbol{\lambda}) - G(\boldsymbol{\lambda}')]\mathbf{u}_{0,N-1} \in \mathscr{Z}(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathscr{X}_0)$$

can be expressed as

$$\mathscr{P}(\lambda, \lambda', \mathscr{X}_0) = \{\mathbf{u} \in \mathbb{R}^{kN} : \Psi(\lambda, \lambda', \mathscr{X}_0)[I - P(\lambda, \lambda')][G(\lambda) - G(\lambda')]\mathbf{u} \leqslant \mu(\lambda, \lambda', \mathscr{X}_0)\}$$

Note that such a set is a polyhedron whose directions are given by $\ker([I - P(\lambda, \lambda')][G(\lambda) - G(\lambda')])$. Then, one may apply condition (5) and conclude that a control sequence $\mathbf{u}_{0,N-1}$ is (N, α, ω)-discerning if and only if

$$\mathbf{u}_{0,N-1} \notin \bigcup_{\lambda, \lambda'} \mathscr{P}(\lambda, \lambda', \mathscr{X}_0)$$

where the union is extended to all the pairs $\lambda, \lambda' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda) \neq r^{\alpha,\omega}(\lambda')$.

3. MODE OBSERVABILITY OVER AN INFINITE HORIZON

In this section, a receding-horizon scheme is proposed for the solution of the active mode-observation problem over an infinite horizon.

The objective is to derive a control sequence $\mathbf{u}_{0,\infty}$ such that, for every $t = N, N+1, \ldots,$ the restriction $\mathbf{u}_{t-N,t-1}$ is an (N, α, ω)-discerning control sequence for system (1) in the interval $[t-N, t]$. A control sequence $\mathbf{u}_{0,\infty}$ that has such a property will be called an (N, α, ω)-*persistently discerning control sequence* for system (1). It should be evident that, if the control sequence $\mathbf{u}_{0,\infty}$ is (N, α, ω)-persistently discerning, then, for every time $t = N, N+1, \ldots,$ the switching sequence $\lambda_{t-N+\alpha,t-\omega}$ in the restricted interval $[t-N+\alpha, t-\omega]$ can be uniquely determined from the observation sequence $\mathbf{y}_{t-N,t}$ in the interval $[t-N, t]$. Thus, such an input sequence makes it possible, at each time t greater than or equal to N, to reconstruct the discrete state up to time $t-\omega$ (with the exception of the first α time instants) on the basis of the information available up to time t.

In [15], conditions for obtaining persistently discerning control sequences were given in the case when no information about the continuous state is available. In this section, we shall focus on the case (quite diffused in practice) in which the set $\mathscr{X}_0$, to which the initial continuous state belongs, is bounded. As a consequence, Assumption A1 has to be replaced by the following:

Assumption A1′
The set $\mathscr{X}_0$ is compact.

3.1. Set-valued estimation of the continuous state

Clearly, a crucial issue in the derivation of a persistently discerning control sequence is how the information on the continuous state (summarized, at the initial time instant $t = 0$, by the set $\mathscr{X}_0$) is propagated over time. Towards this end, in this section some basic concepts on recursive *set-valued estimation* of the continuous state are briefly recalled (the interested reader is referred to [16, 19, 20]).

Let us suppose, preliminarily, that at each time instant $t = 0, 1, \ldots$ the whole switching sequence $\lambda_{0,t}$ is available. This hypothesis is instrumental to present the set-valued estimation procedure in a (hopefully) clear way and will be removed in Section 3.2.

Moreover, let us denote by $\mathscr{X}_t(\lambda_{0,t-1})$ the *feasible states set* at time t, i.e. the set of all the continuous states x_t that are consistent with the observation sequence $\mathbf{y}_{0,t-1}$, with the control sequence $\mathbf{u}_{0,t-1}$, and with the switching sequence $\lambda_{0,t-1}$.

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In order to compute the set $\mathcal{X}_t(\boldsymbol{\lambda}_{0,t-1})$, let us consider the *innovation set* $\mathcal{I}_t(\lambda_t)$ at time t, i.e. the set of all the continuous states x_t that are consistent with the observation y_t and with the discrete state λ_t. It is immediate to see that

$$\mathcal{I}_t(\lambda_t)=\{x\in\mathbb{R}^n:\exists v\in\mathcal{V} \text{ with } C(\lambda_t)x+v=y_t\}$$

By exploiting such a definition, the sets $\mathcal{X}_t(\boldsymbol{\lambda}_{0,t-1})$ for $t=1,2,\ldots$ can be computed recursively by means of a simple two-steps procedure.

Procedure 1
At any time $t=0,1,\ldots$

1. [*innovation*] perform the intersection between the feasible state sets $\mathcal{X}_t(\boldsymbol{\lambda}_{0,t-1})$ and the innovation set $\mathcal{I}_t(\lambda_t)$

$$\mathcal{X}_t^+(\boldsymbol{\lambda}_{0,t})=\mathcal{X}_t(\boldsymbol{\lambda}_{0,t-1})\cap\mathcal{I}_t(\lambda_t)$$

2. [*propagation*] propagate the set $\mathcal{X}_t^+$ to the next time instant $t+1$ by means of the state equation (1a)

$$\mathcal{X}_{t+1}(\boldsymbol{\lambda}_{0,t})=[A(\lambda_t)\mathcal{X}_t^+(\boldsymbol{\lambda}_{0,t})]\oplus\{B(\lambda_t)u_t\}\oplus\mathcal{W}$$

The recursion is initialized at time $t=0$ with $\mathcal{X}_0(\emptyset)=\mathcal{X}_0$ (here $\emptyset$ denotes an empty sequence).

It is worth noting that the computations of the sets $\mathcal{X}_t^+(\boldsymbol{\lambda}_{0,t})$ and $\mathcal{X}_{t+1}(\boldsymbol{\lambda}_{0,t})$ in the innovation and propagation steps, respectively, of the set-valued estimation algorithm can be carried out exactly only in some special cases. For example, this happens when the three sets $\mathcal{X}_0$, $\mathcal{W}$, and $\mathcal{V}$ are polytopes. In this case, the feasible states set $\mathcal{X}_t(\boldsymbol{\lambda}_{0,t-1})$ also turns out to be a polytope for every $t=0,1,\ldots$.

In general, however, the exact computation of the feasible states sets $\mathcal{X}_t(\boldsymbol{\lambda}_{0,t-1})$ may be a computationally quite demanding (or even impossible) task. Thus, usually only an outer approximation of the set $\mathcal{X}_t(\boldsymbol{\lambda}_{0,t-1})$ is determined inside a class of sets of fixed complexity (e.g. ellipsoids, boxes, or parallelotopes. See [17, 19, 21]). It should be evident that, in this case, in order to reduce the conservatism such an outer approximation has to be chosen according to some optimality criterion (e.g. with minimum volume). Further, an outer approximation of fixed complexity is also usually adopted in the polytopic case in order to avoid an uncontrolled growth of the number of vertices associated with the polytopic set $\mathcal{X}_{t+1}(\boldsymbol{\lambda}_{0,t})$, which would make the algorithm eventually unfeasible from the computational point of view (see. e.g. [22]).

Then, instead of resorting to the optimal set-valued estimation algorithm described in Procedure 1 we suppose that some approximate algorithm is available that provides an *estimation set* $\hat{\mathcal{X}}_t(\boldsymbol{\lambda}_{0,t-1})$ containing the feasible states set, i.e. such that

$$\hat{\mathcal{X}}_t(\boldsymbol{\lambda}_{0,t-1})\supseteq\mathcal{X}_t(\boldsymbol{\lambda}_{0,t-1})$$

As to the requirements of our application, the need is that the estimation set does not grow unbounded over time. Then, the following assumption is needed:

Assumption A3
The approximate set-valued estimation algorithm is such that for any time t and for any switching sequence $\boldsymbol{\lambda}_{0,t-1}$ the Lebesgue measure of the estimation set $\hat{\mathscr{X}}_t(\boldsymbol{\lambda}_{0,t-1})$ is bounded above by a constant M.

Assumption A3 can be seen as a sort of minimal requirement that a sensible set-valued estimation algorithm should satisfy. Of course, under suitable observability conditions, such an assumption holds true for the optimal set-valued estimation algorithm described in Procedure 1. This observation is summarized in the following well-known result:

Lemma 1
Suppose that assumptions A1′ and A2 are satisfied. Furthermore, suppose that system (1) is uniformly observable with respect to the continuous state x_t, i.e. there exists an integer N_o such that for any switching sequence $\boldsymbol{\lambda}_{0,N_o-1}$ the observability matrix $F(\boldsymbol{\lambda}_{0,N_o-1})$ has full rank. Then the Lebesgue measure of the feasible states set $\mathscr{X}_t(\boldsymbol{\lambda}_{0,t-1})$ is bounded above by a constant M.

It is immediate to verify that a similar result holds even if, at each step of Procedure 1, an approximation error is committed in the computation of the set $\mathscr{X}_t(\boldsymbol{\lambda}_{0,t-1})$, provided that such an error is uniformly bounded by some constant (in fact, it would play the same role of a bounded measurement noise).

3.2. Moving-horizon observation of the switching sequence

As to the observation of the switching sequence, in order to exploit the results developed in Section 2 in a finite-horizon framework, we shall adopt a moving-horizon approach.

Towards this end, let us denote by $\mathscr{F}_{0,t|t} \subseteq \mathscr{L}^{t+1}$ the *set of feasible switching sequences* at time t. As will be clear in the following, such a set summarizes all the information that has been acquired on the whole switching sequence $\boldsymbol{\lambda}_{0,t}$ up to time t. Moreover, given two time instants t_1 and t_2 with $t_1 \leqslant t_2 \leqslant t$, let us denote by $\mathscr{F}_{t_1,t_2|t} \subseteq \mathscr{L}^{t_2-t_1+1}$ the set obtained by considering the restriction of each sequence in $\mathscr{F}_{0,t|t}$ to the interval $[t_1, t_2]$.

The set of feasible switching sequences $\mathscr{F}_{0,t|t} \subseteq \mathscr{L}^{t+1}$ can be computed recursively by means of the following moving-horizon procedure:

Procedure 2
At any time, $t = N, N+1, \ldots$

1. compute the set $\mathscr{C}_t$ of all the switching sequences in the interval $[t-N, t]$, which are consistent with the observation sequence $\mathbf{y}_{t-N,t}$, the control sequence $\mathbf{u}_{t-N,t-1}$ and the available information on the continuous state x_{t-N};
2. merge the novel information provided by $\mathscr{C}_t$ with the old one provided by $\mathscr{F}_{0,t-1|t-1}$ to obtain the new feasible switching sequences set $\mathscr{F}_{0,t|t}$.

The recursion is initialized by setting $\mathscr{F}_{0,t|t} = \mathscr{L}^{t+1}$ for $t = 0, 1, \ldots, N-1$.

Let us now consider the computation of the set $\mathscr{C}_t$ in the first step of the procedure. If the switching sequence up to time $t-N$ were exactly known, then the set to which the continuous state

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x_{t-N} is known to belong would be the feasible state set $\hat{\mathscr{X}}_{t-N}(\boldsymbol{\lambda}_{0,t-N-1})$. However, in general, it is not possible to reconstruct exactly the whole switching sequence $\boldsymbol{\lambda}_{0,t-N-1}$. Therefore the continuous state x_{t-N} is only known to belong to the finite union

$$\hat{\mathscr{X}}_{t-N|t-1} \triangleq \bigcup_{\boldsymbol{\lambda}\in\mathscr{F}_{0,t-N-1|t-1}} \hat{\mathscr{X}}_{t-N}(\boldsymbol{\lambda})$$

On the basis of such a consideration, it is immediate to see that the set $\mathscr{C}_t$ can be computed as

$$\mathscr{C}_t = \{\boldsymbol{\lambda}\in\mathscr{L}^{N+1} : \mathbf{y}_{t-N,t} \in \mathscr{Y}(\boldsymbol{\lambda}, \mathbf{u}_{t-N,t-1}, \hat{\mathscr{X}}_{t-N|t-1})\}$$

In general, determining the set $\mathscr{C}_t$ can be a very hard task. However, if the sets $\mathscr{W}$ and $\mathscr{V}$ and $\hat{\mathscr{X}}_{t-N|t-1}$ are polytopes, then such procedure results to be computationally tractable (see [12] for a similar case).

3.3. A receding-horizon active mode-observation scheme

In virtue of the two procedures developed in Sections 3.1 and 3.2, we are now able to build a receding-horizon scheme for the active observation of the switching sequence (similar to the one proposed in [15] where the sets $\hat{\mathscr{X}}_{t-N|t-1}$ were disregarded).

Towards this end, note that the control u_t, to be chosen at time t, influences the observability of the discrete state in all the intervals of length $N+1$ containing the instant t, i.e. in all the intervals

$$[t-N+1+i, t+1+i], \quad i=0,1,\ldots,N-1$$

when $t \geqslant N-1$, and

$$[j, j+N], \quad j=0,1,\ldots,t$$

when $t<N-1$. Such intervals can be equivalently denoted for any $t=0,1,\ldots$ by

$$[t-N+1+i, t+1+i], \quad i=i_{0|t}, i_{0|t}+1,\ldots,N-1$$

where $i_{0|t} \triangleq \max\{0, N-1-t\}$. The definition of $i_{0|t}$ will be useful in the following to make the formulas more compact. By using such a definition, in order to satisfy the mode observability conditions in all the intervals, at any time $t=0,1,\ldots,$ the control u_t has to be chosen so that

$$[G(\boldsymbol{\lambda})-G(\boldsymbol{\lambda}')]\mathbf{u}_{t-N+1+i,t+i} \notin \mathscr{Q}(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \hat{\mathscr{X}}_{t-N+1+i}(\boldsymbol{\lambda}'')) \tag{11}$$

for every $\boldsymbol{\lambda},\boldsymbol{\lambda}'\in\mathscr{F}_{t-N+1+i,t+i|t}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, for every $\boldsymbol{\lambda}''\in\mathscr{F}_{0,t-N+i|t}$, and for every $i=i_{0|t}, i_{0|t}+1,\ldots,N-1$.

Note that such conditions depend on the last $N-1$ controls $\mathbf{u}_{t-N+1,t-1}$ (that have already been applied at time t) and on the control sequence $\mathbf{u}_{t,t+N-1}$ (which is not yet fixed at time t), composed by the control u_t (to be chosen at the current time) and by the future $N-1$ controls $\mathbf{u}_{t+1,t+N-1}$. With this respect, suitable matrices $R_i^-(\boldsymbol{\lambda},\boldsymbol{\lambda}')$ and $R_i^+(\boldsymbol{\lambda},\boldsymbol{\lambda}')$ can be defined such that each condition in (11) can be rewritten as

$$R_i^-(\boldsymbol{\lambda},\boldsymbol{\lambda}')\mathbf{u}_{t-N+1,t-1}+R_i^+(\boldsymbol{\lambda},\boldsymbol{\lambda}')\mathbf{u}_{t,t+N-1}\notin\mathscr{Q}(\boldsymbol{\lambda},\boldsymbol{\lambda}',\hat{\mathscr{X}}_{t-N+1+i}(\boldsymbol{\lambda}''))$$

More specifically, each matrix $R_i^-(\boldsymbol{\lambda},\boldsymbol{\lambda}')$ is obtained as the juxtaposition of a $(N+1)m\times ik$ matrix of zeros with the first $(N-1-i)k$ columns of $[G(\boldsymbol{\lambda})-G(\boldsymbol{\lambda}')]$. Similarly, each matrix $R_i^+(\boldsymbol{\lambda},\boldsymbol{\lambda}')$ is obtained as the juxtaposition of the last $(i+1)k$ columns of $[G(\boldsymbol{\lambda})-G(\boldsymbol{\lambda}')]$ with an $(N+1)m\times(N-1-i)k$ matrix of zeros.

As to the control sequence $\mathbf{u}_{t,t+N-1}$, one may adopt the following strategy: first, determine a control sequence $\mathbf{u}_{t,t+N-1|t}=\mathrm{col}(u_{t|t},u_{t+1|t},\ldots,u_{t+N-1|t})$ of length N that satisfies all the mode observability constraints; then, actually apply only the first control of such a sequence. The same mechanism can be applied stage after stage. Then the proposed strategy gives rise to the following iterative receding-horizon procedure:

Procedure 3
For any $t=0,1,\ldots$

1. find a control sequence $\mathbf{u}_{t,t+N-1|t}$ such that
$$R_i^-(\boldsymbol{\lambda},\boldsymbol{\lambda}')\mathbf{u}_{t-N+1,t-1}+R_i^+(\boldsymbol{\lambda},\boldsymbol{\lambda}')\mathbf{u}_{t,t+N-1|t}\notin\mathscr{Q}(\boldsymbol{\lambda},\boldsymbol{\lambda}',\hat{\mathscr{X}}_{t-N+1+i}(\boldsymbol{\lambda}''))\quad(12)$$
for every $\boldsymbol{\lambda},\boldsymbol{\lambda}'\in\mathscr{F}_{t-N+1+i,t+i|t}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda})\neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, for every $\boldsymbol{\lambda}''\in\mathscr{F}_{0,t-N+i|t}$, and for every $i=i_{0|t},i_{0|t}+1,\ldots,N-1$;
2. apply the first control of $\mathbf{u}_{t,t+N-1|t}$, i.e. set $u_t=u_{t|t}$.

In the following, we shall call *feasible* the control sequences $\mathbf{u}_{t,t+N-1|t}$ that satisfy all the conditions in step 1 of Procedure 3. It is worth noting that Procedure 3 does not lead to the choice of a *unique* control sequence $\mathbf{u}_{t,t+N-1|t}$; on the contrary it just gives a certain number of conditions that the control sequence $\mathbf{u}_{t,t+N-1|t}$ has to satisfy in order to ensure the observability of the discrete state. As a consequence, such a procedure is well suited to take into account other possible control objectives. For example, the proposed iterative scheme may be used in connection with a receding-horizon model predictive control scheme, where each control action is generated by solving an open-loop optimal control problem over a finite horizon (see, for an introduction, [23]). In this case, conditions (12) can be seen as constraints on the control sequence to be applied whenever the observability of the discrete state is required.

As, in our context, we are supposing to choose a persistently discerning control sequence, then at a generic $t=N,N+1,\ldots$ the knowledge of the discrete states in the interval $[\alpha,t-\omega]$ is guaranteed. As a consequence, the cardinality of the set $\mathscr{F}_{0,t|t}$ remains bounded with t. Thus, the number of sets $\hat{\mathscr{X}}_t(\lambda)$ to be considered in step 1 of Procedure 1 remains bounded, and the same holds for the propagation (step 2 in Procedure 1). This ensures the computational tractability of the problem over an infinite horizon.

The following theorem ensures the solvability of step 2 of Procedure 3 at every time stage and then the possibility of applying it to obtain an (N,α,ω)-persistently discerning control sequence.

Theorem 3
Suppose that Assumptions A1′, A2, and A3 hold. Furthermore, suppose that for every $\boldsymbol{\lambda},\boldsymbol{\lambda}'\in\mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda})\neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, we have

$$\mathrm{rank}[G(\boldsymbol{\lambda})-G(\boldsymbol{\lambda}')]>0\quad(13)$$

If Procedure 3 is applied iteratively to generate the control vectors $u_t, t=0,1,\ldots,$ then

(i) at every time $t=0,1,\ldots,$ feasible control sequences $\mathbf{u}_{t,t+N-1|t}$ exist in almost all directions;
(ii) there exists a constant K (independent of time) such that, at every time $t=0,1,\ldots,$ one can always find a feasible control sequence $\mathbf{u}_{t,t+N-1|t}$ with $\|\mathbf{u}_{t,t+N-1|t}\|\leqslant K$.

It is immediate to verify that condition (13) ensures the existence of infinite many control sequences $\mathbf{u}'_{0,N-1}$ satisfying

$$[G(\lambda)-G(\lambda')]\mathbf{u}'_{0,N-1}\neq 0$$

for every $\lambda,\lambda'\in\mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda)\neq r^{\alpha,\omega}(\lambda')$. This is required to ensure the solvability of condition (11) regardless of the form of the sets $\mathscr{Q}(\lambda,\lambda',\hat{\mathscr{X}}_{t-N+1+i}(\lambda''))$.

As to point (ii) of Theorem 3, note that, in the limit case wherein the system is noise-free, each set $\hat{\mathscr{X}}_{t-N+1+i}(\lambda'')$ is a singleton provided that uniform observability of the continuous state holds. This, in turn, also implies that each set $\mathscr{Q}(\lambda,\lambda',\hat{\mathscr{X}}_{t-N+1+i}(\lambda''))$ is a singleton. It is easy to verify that in this case the constant K can be made arbitrarily close to zero, i.e. one can ensure the observability of the discrete state by applying arbitrarily small controls. More generally, the constant K increases with the amplitudes of the sets $\mathscr{W}$ and $\mathscr{V}$, i.e. the more noisy the system, the larger are the values of the controls that have to be applied so as to ensure mode observation.

4. NUMERICAL EXAMPLE

Consider the switching system described by the Equations (1) with

$$A(1)=\begin{bmatrix}0.97 & 0.16\\ -0.33 & 0.64\end{bmatrix},\quad A(2)=\begin{bmatrix}0.98 & 0.18\\ -0.18 & 0.8\end{bmatrix},\quad B(1)=\begin{bmatrix}0.02\\ 0.16\end{bmatrix},\quad B(2)=\begin{bmatrix}0.01\\ 0.09\end{bmatrix}$$

$$C(1)=C(2)=[0\ \ 1]$$

The system results from the discretization of two driven continuous time damped oscillators with different damping coefficients (0.5 and 1, respectively).The sample time is set to 0.2 s.

Suppose that the initial continuous state x_0 and the noises $w_t,\ v_t,\ t=0,1,\ldots$ belong to the polytopic compact sets $\mathscr{X}_0=[-1,1]^2$, $\mathscr{W}=[-0.01,0.01]^2$, and $\mathscr{V}=[-0.01,0.01]$, respectively.

By choosing $\alpha=0$, $\omega=1$, and $N=2$, one can satisfy conditions (10). Therefore, since the initial state is known to belong to a known compact set, such a system turns out to be $(2,0,1)$-mode observable. On the contrary, it can be easily verified that such a system would not be $(2,0,1)$-mode observable if no *a priori* information were available on the initial condition, i.e. if $\mathscr{X}_0=\mathbb{R}^2$. With simple calculations it is possible to verify that the proposed switching system is uniformly observable with respect to the continuous state. Moreover, since the initial state and the noises belong to polytopic sets, the optimal set-valued estimation algorithm described in Procedure 1 can be applied and therefore Assumption A3 holds true (see Lemma 1). Since all the assumptions of Theorem 3 are met, we conclude that Procedure 3 can be applied to generate $(2,0,1)$-persistently discerning control sequences $\mathbf{u}_{0,\infty}$.

To simulate the system and to apply the receding-horizon scheme the following parameters have been used: the initial state has been set to $[0,0]^\top$ and the noises sequences have been generated according to uniform distributions defined in their respective polytopic domains. The switching sequence is depicted in Figure 2.

In Figure 3 the geometric conditions (relative to some time instants) that must be satisfied by a control sequence in order to be persistently discerning are shown. In particular, for this particular example, the constant control sequence $u_t = 10, t = 0, 1, \ldots$ turns out to be persistently discerning.

In Figure 4 some samples from the evolution of the polytopes $\mathscr{X}_t^+(\boldsymbol{\lambda}_{0,t})$ are provided. Note that these polytopes are instrumental to compute the regions depicted in Figure 3 and consequently to apply Procedure 3.

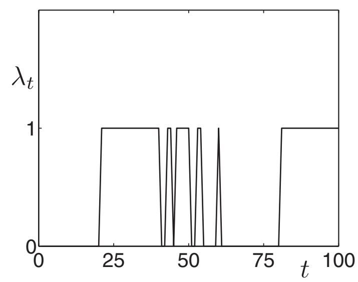

Figure 2. Evolution of the system mode.

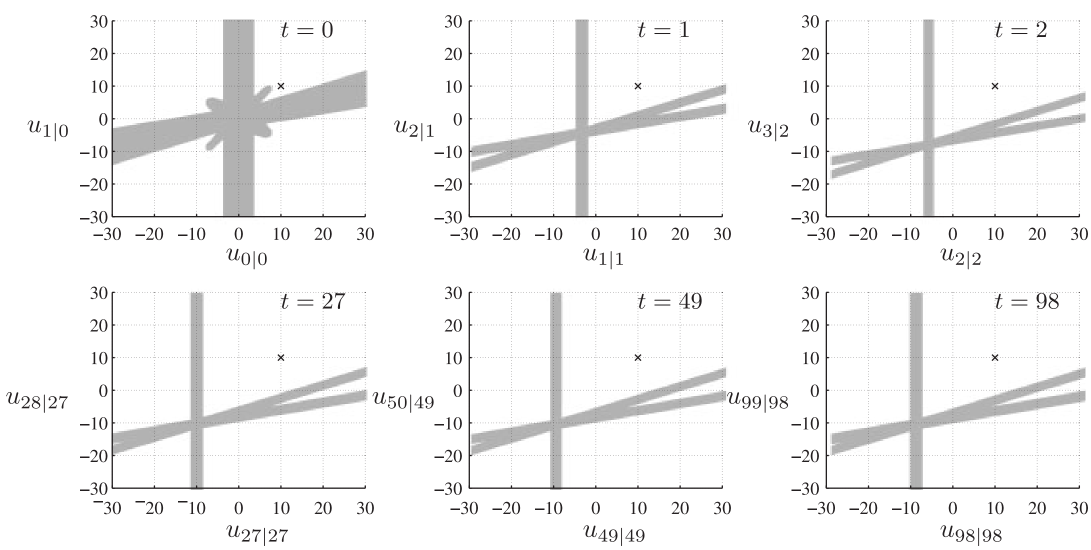

Figure 3. Graphical representation of the mode observability conditions on the control sequence $\mathbf{u}_{t,t+1|t}$ (see step 1 of Procedure 3) for different time instants. The grey regions correspond to the sets of the non-discerning control sequences. The cross indicates the applied control sequence.

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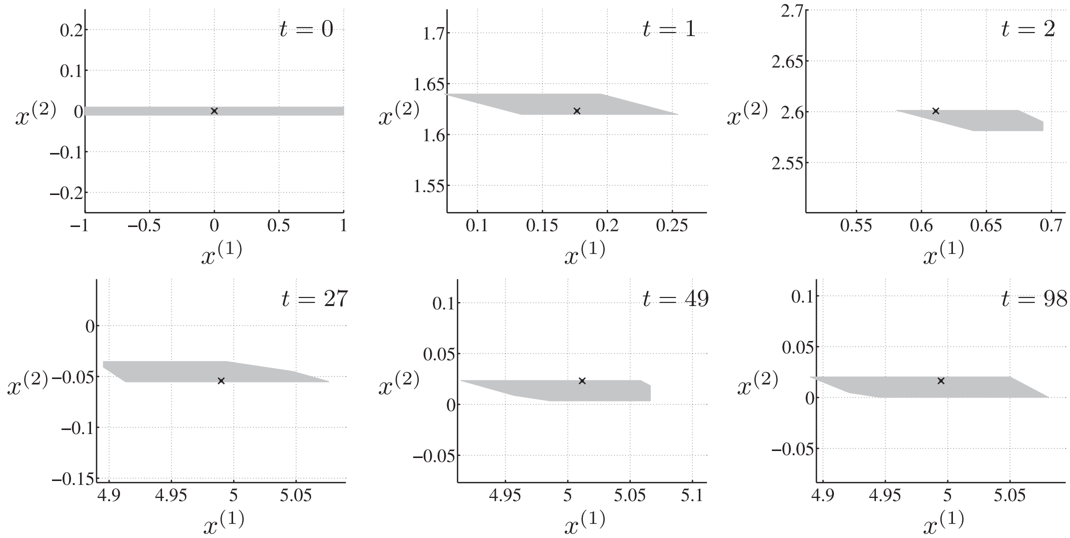

Figure 4. Graphical representation of the evolution of the polytopes $\mathscr{X}_t^{+}(\boldsymbol{\lambda}_{0,t})$. The cross represents the actual state of the system.

5. CONCLUSIONS

In this paper the active mode-observation problem is addressed for a class of discrete-time linear systems in a finite and infinite-horizon setting. The main contribution is the generalization of the results in [15] in order to take into account possible *a priori* information on the continuous state. Conditions that characterize discerning controls in a finite-horizon setting are obtained and a procedure is proposed, in order to derive 'persistently discerning' control sequences over an infinite horizon. To take into account (potentially useful) information on the state of the system, the resulting scheme combines the observation of the switching sequence with a recursive set-valued estimation algorithm. It is worth noting that, whenever the identification of the switching pattern is just one among several control objectives, the proposed procedure could be applied in connection with a receding-horizon model predictive control scheme. Further studies in this direction would be of great interest.

APPENDIX

Proof of Theorem 1

First note that, by definition, a control sequence $\mathbf{u}_{0,N-1}$ is (N,α,ω)-discerning if and only if, for every pair of switching sequences $\boldsymbol{\lambda},\boldsymbol{\lambda}'\in\mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda})\neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, we have

$$F(\boldsymbol{\lambda})x+G(\boldsymbol{\lambda})\mathbf{u}_{0,N-1}+H(\boldsymbol{\lambda})\mathbf{w}+\mathbf{v}\neq F(\boldsymbol{\lambda}')x'+G(\boldsymbol{\lambda}')\mathbf{u}_{0,N-1}+H(\boldsymbol{\lambda}')\mathbf{w}'+\mathbf{v}' \tag{A1}$$

for every pair of initial states $x,x'\in\mathscr{X}_0$, for every pair of system noise sequences $\mathbf{w},\mathbf{w}'\in\mathscr{W}^N$, and for every pair of measurement noise sequences $\mathbf{v},\mathbf{v}'\in\mathscr{V}^{N+1}$.

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Under Assumption A1, the generic initial state x can be decomposed as the sum of a vector $s \in \mathscr{S}$ and a bounded vector $b \in \mathscr{B}$. Similarly, x' can be written as $x' = s' + b'$ for some $s' \in \mathscr{S}$ and $b' \in \mathscr{B}$. As a consequence, condition (A1) can be rewritten in the equivalent form

$$[-F(\lambda)|F(\lambda')]\begin{bmatrix} s \\ s' \end{bmatrix} \neq [G(\lambda)-G(\lambda')]\mathbf{u}_{0,N-1}+F(\lambda)b-F(\lambda')b'+H(\lambda)\mathbf{w}+\mathbf{v}-H(\lambda')\mathbf{w}'-\mathbf{v}' \quad \text{(A2)}$$

Since s and s' are generic vectors belonging to the linear subspace $\mathscr{S}$, the left-hand side is a generic vector belonging to the linear subspace $\{F(\lambda)\mathscr{S}\} \oplus \{F(\lambda')\mathscr{S}\}$. Thus, condition (A2) is satisfied if and only if the right-hand side does *not* belong to such a linear subspace or, equivalently, if and only if the projection of the right-hand side on the subspace orthogonal to $\{F(\lambda)\mathscr{S}\} \oplus \{F(\lambda')\mathscr{S}\}$ is not null, that is,

$$[I-P(\lambda,\lambda')]\{[G(\lambda)-G(\lambda')]\mathbf{u}_{0,N-1}+F(\lambda)b-F(\lambda')b'+H(\lambda)\mathbf{w}+\mathbf{v}-H(\lambda')\mathbf{w}'-\mathbf{v}'\} \neq 0 \quad \text{(A3)}$$

With $\mathscr{Q}(\lambda,\lambda',\mathscr{X}_0)$ defined as in the statement of the lemma, it is immediate to see that (A3) turns out to be equivalent to (5). Note that the boundedness of the sets $\mathscr{B}$, $\mathscr{W}$, and $\mathscr{V}$ ensures that the set $\mathscr{Q}(\lambda,\lambda',\mathscr{X}_0)$ is also bounded. □

Proof of Corollary 1

Consider a pair of switching sequences $\lambda, \lambda' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda) \neq r^{\alpha,\omega}(\lambda')$. Then, two possibilities may arise:

(a) $[I-P(\lambda,\lambda')][G(\lambda)-G(\lambda')]\mathbf{u}'_{0,N-1}$ is equal to 0. In this case, by hypothesis, $0 \notin \mathscr{Q}(\lambda,\lambda',\mathscr{X}_0)$ and condition

$$g[I-P(\lambda,\lambda')][G(\lambda)-G(\lambda')]\mathbf{u}'_{0,N-1} \notin \mathscr{Q}(\lambda,\lambda',\mathscr{X}_0) \quad \text{(A4)}$$

is automatically verified for every $g \in \mathbb{R}$.

(b) $[I-P(\lambda,\lambda')][G(\lambda)-G(\lambda')]\mathbf{u}'_{0,N-1}$ is not equal to 0. In this case, the boundedness of $\mathscr{Q}(\lambda,\lambda',\mathscr{X}_0)$ ensures that (A4) holds for every g outside some bounded interval $\mathscr{I}(\lambda,\lambda',\mathbf{u}'_{0,N-1})$ (note that $\mathscr{I}(\lambda,\lambda',\mathbf{u}'_{0,N-1})$ might even be empty).

Then, in order to satisfy condition (5) for every $\lambda, \lambda' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda) \neq r^{\alpha,\omega}(\lambda')$, it is sufficient to choose the scalar parameter g so that

$$g \notin \bigcup_{\lambda,\lambda'} \mathscr{I}(\lambda,\lambda',\mathbf{u}'_{0,N-1})$$

where the union is extended to every $\lambda, \lambda' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda) \neq r^{\alpha,\omega}(\lambda')$ and $[I-P(\lambda,\lambda')][G(\lambda)-G(\lambda')]\mathbf{u}'_{0,N-1} \neq 0$. □

Proof of Theorem 2 (*sufficiency*)

Let us denote by $\mathscr{K}(\lambda,\lambda')$ the kernel of the matrix $[I-P(\lambda,\lambda')][G(\lambda)-G(\lambda')]$. In the light of Corollary 1, a sufficient condition for the (N,α,ω)-mode observability of the system is that there exists a control sequence $\mathbf{u}'_{0,N-1}$ that does *not* belong to the set

$$\mathscr{K}_N^{\alpha,\omega} \triangleq \bigcup_{\lambda,\lambda'} \mathscr{K}(\lambda,\lambda')$$

where the union is extended to every $\lambda, \lambda' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda) \neq r^{\alpha,\omega}(\lambda')$ and $0 \in \mathscr{Q}(\lambda,\lambda',\mathscr{X}_0)$.

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Suppose now that

$$[I-P(\lambda,\lambda')][G(\lambda)-G(\lambda')]\neq 0 \tag{A5}$$

for every $\lambda,\lambda'\in\mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda)\neq r^{\alpha,\omega}(\lambda')$ and $0\in\mathscr{Z}(\lambda,\lambda',\mathscr{X}_0)$. It is immediate to see that, whenever condition (A5) is satisfied, each kernel $\mathscr{K}(\lambda,\lambda')$ is a null set (i.e. a set with a null Lebesgue measure). As a consequence, also $\mathscr{K}_N^{\alpha,\omega}$ is a null set, since it is obtained as the union of a finite number of null sets. Therefore, in this case, condition $\mathbf{u}'_{0,N-1}\notin\mathscr{K}_N^{\alpha,\omega}$ (and consequently condition (8)) is satisfied for almost every choice of the control sequence $\mathbf{u}'_{0,N-1}$.

Thus in order to conclude the first part of the proof it is sufficient to show that condition (9) is equivalent to (A5). Towards this end, recall that $[I-P(\lambda,\lambda')]$ is the matrix of the orthogonal projection on the subspace orthogonal to $\{F(\lambda)\mathscr{S}\}\oplus\{F(\lambda')\mathscr{S}\}=\text{span}([F(\lambda)S|F(\lambda')S])$. Hence, it is immediate to verify the equivalence of conditions (9) and (A5), in that condition (9) ensures that there exists at least one column of $[G(\lambda)-G(\lambda')]$ that does not belong to $\text{span}([F(\lambda)S|F(\lambda')S])$ and, hence, with a non-null projection on the subspace orthogonal to $\text{span}([F(\lambda)S|F(\lambda')S])$.

(*necessity*) First note that a control sequence $\mathbf{u}_{0,N-1}$ can be (N,α,ω)-discerning only if

$$[I-P(\lambda,\lambda')][G(\lambda)-G(\lambda')]\mathbf{u}_{0,N-1}\neq 0 \tag{A6}$$

for every $\lambda,\lambda'\in\mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda)\neq r^{\alpha,\omega}(\lambda')$ and $0\in\mathscr{Z}(\lambda,\lambda',\mathscr{X}_0)$. Then the proof is concluded by noting that such a condition can be satisfied only if (A5) holds and by recalling that (A5) is equivalent to (9). □

Proof of Theorem 3

The proof can be given by induction. First note that, at the initial time $t=0$, condition (13) ensures the (N,α,ω)-mode observability of system (1) (see Remark 1). Then it is possible to find a control sequence $\mathbf{u}_{0,N-1|0}$ satisfying all the observability conditions (12).

Suppose now that, at time $t-1$ (with $t\geqslant 1$), there exists a control sequence $\mathbf{u}_{t-1,t+N-2|t-1}$ that satisfies all the conditions of step 2 of Procedure 3. Moreover, suppose that u_{t-1} is chosen as the first control of such a sequence. Then, we need to show that, also at time t, there exists a control sequence $\mathbf{u}_{t,t+N-1|t}$ satisfying all the observability conditions (12). With this respect, for each condition in (12), two possibilities have to be considered:

(a) The matrix $R_i^+(\lambda,\lambda')$ is equal to 0 (note that this may happen only for $i<N-1$, in that for $i=N-1$ we have $R_i^+(\lambda,\lambda')=G(\lambda)-G(\lambda')$, which is different from 0 under condition (13)). Clearly, in this case, the corresponding condition *does not* depend on the control sequence $\mathbf{u}_{t,t+N-1|t}$. As a consequence, by the induction hypotheses, the past controls $\mathbf{u}_{t-N+1,t-1}$ have certainly been chosen in such a way that $R_i^-(\lambda,\lambda')\mathbf{u}_{t-N+1,t-1}\notin\mathscr{Z}(\lambda,\lambda',\hat{\mathscr{X}}_{t-N+1+i}(\lambda''))$ for every $\lambda''\in\mathscr{F}_{0,t-N+i|t}$ (otherwise $\mathbf{u}_{t-1,t+N-2|t-1}$ would have not been feasible).

(b) The matrix $R_i^+(\lambda,\lambda')$ is different from 0. In this case, the corresponding condition depends on the future controls, and the sequence $\mathbf{u}_{t,t+N-1|t}$ has to be chosen so that

$$R_i^+(\lambda,\lambda')\mathbf{u}_{t,t+N-1|t}\notin\mathscr{Z}(\lambda,\lambda',\hat{\mathscr{X}}_{t-N+1+i}(\lambda''))-R_i^-(\lambda,\lambda')\mathbf{u}_{t-N+1,t-1} \tag{A7}$$

Then, one can always find a feasible control sequence by means of the following simple procedure (similar to the one adopted in Corollary 1): determine a control sequence $\mathbf{u}'_{t,t+N-1|t}$ satisfying

$$R_i^+(\lambda,\lambda')\mathbf{u}'_{t,t+N-1|t}\neq 0 \tag{A8}$$

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for every $i = i_{0|t}, i_{0|t}+1, \ldots, N-1$ and for every $\lambda, \lambda' \in \mathscr{L}^{N+1}$ with $r^{\alpha,\omega}(\lambda) \neq r^{\alpha,\omega}(\lambda')$ and $R_i^+(\lambda, \lambda') \neq 0$; scale the control sequence $\mathbf{u}'_{t,t+N-1|t}$ by means of a suitable scalar g. Note that each of the kernels of the matrices $R_i^+(\lambda, \lambda')$ is a null set; hence, the proposed construction can be applied for almost every choice of the vector $\mathbf{u}'_{t,t+N-1|t}$ (thus proving fact (i)). Furthermore, Assumption A3 ensures that the Lebesgue measure of the sets $\mathscr{Q}(\lambda, \lambda', \hat{\mathscr{X}}_{t-N+1+i}(\lambda''))$ is uniformly bounded by some constant Q. Then, the existence of infinite many choices of the scale factor g making the control sequence $g\mathbf{u}'_{t,t+N-1|t}$ feasible can be shown following the same lines of the proof of Corollary 1.

Fact (ii) descends directly from Assumption A3 and from the fact that the bounding constant Q does not depend on the time t. The proof follows from standard arguments (see, e.g. the proof of Theorem 4 in [15]) and is omitted. □

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