Automatica 45 (2009) 2557–2562

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Synchronization in networks of identical linear systems☆

Luca Scardovi[a,*], Rodolphe Sepulchre[b]

[a] Department of Mechanical and Aerospace Engineering, Princeton University, USA

[b] Department of Electrical Engineering and Computer Science, University of Liège, Belgium

ARTICLE INFO

Article history:
Received 22 April 2008
Received in revised form
14 March 2009
Accepted 1 July 2009
Available online 19 August 2009

Keywords:
Synchronization
Consensus
Multi-agent systems
Weak coupling
Oscillator analysis

ABSTRACT

The paper investigates the synchronization of a network of identical linear state-space models under a possibly time-varying and directed interconnection structure. The main result is the construction of a dynamic output feedback coupling that achieves synchronization if the decoupled systems have no exponentially unstable mode and if the communication graph is uniformly connected. The result can be interpreted as a generalization of classical consensus algorithms. Stronger conditions are shown to be sufficient – but to some extent, also necessary – to ensure synchronization with the diffusive static output coupling often considered in the literature.

1. Introduction

In recent years, consensus, coordination and synchronization problems have been popular subjects in systems and control, motivated by many applications in physics, biology, and engineering. These problems arise in multi-agent systems with the collective objective of reaching agreement about some variables of interest.

In the *consensus* literature, the emphasis is on the communication constraints rather than on the individual dynamics: the agents exchange information according to a communication graph that is not necessarily complete, nor even symmetric or time-invariant, but, in the absence of communication, the agreement variables usually have no dynamics. It is the exchange of information only that determines the time-evolution of the variables, aiming at asymptotic synchronization to a common value. The convergence of such consensus algorithms has attracted much attention in recent years. It only requires a weak form of connectivity for the communication graph (Jadbabaie, Lin, & Morse, 2003; Moreau, 2005, 2004; Olfati-Saber & Murray, 2004).

In the *synchronization* literature, the emphasis is on the individual dynamics rather than on the communication limitations: the communication graph is often assumed to be complete (or all-to-all), but in the absence of communication, the time-evolution of the systems' variables can be oscillatory or even chaotic. The system dynamics can be modified through the information exchange, and, as in the consensus problem, the goal of the interconnection is to reach synchronization to a common solution of the individual dynamics (Hale, 1996; Pham & Slotine, 2007; Pogromsky, 1998; Stan & Sepulchre, 2007).

Coordination problems encountered in the engineering world can often be rephrased as consensus or synchronization problems in which both the individual dynamics and the limited communication aspects play an important role. Designing interconnection control laws that can ensure synchronization of relevant variables is therefore a control problem that has attracted quite some attention in recent years (Nair & Leonard, 2008; Sarlette, Sepulchre, & Leonard, 2007; Scardovi, Leonard, & Sepulchre, 2008; Scardovi, Sarlette, & Sepulchre, 2007; Sepulchre, Paley, & Leonard, 2008).

The present paper deals with a fairly general solution of the synchronization problem in the linear case. Assuming N identical individual agents dynamics each described by the linear state-space model (A, B, C), the main result is the construction of a dynamic output feedback controller that ensures exponential synchronization to a solution of the linear system $\dot{x} = Ax$ under the following assumptions: (i) A has no exponentially unstable modes, (ii) (A, B) is stabilizable and (A, C) is detectable, and (iii) the communication graph is uniformly connected. The result can be interpreted as a generalization of classical consensus algorithms, studied recently, corresponding to the particular case

☆ The work is supported in part by ONR grants N00014–02–1–0826 and N00014–04–1–0534. The material in this paper was partially presented at 47th IEEE Conference on Decision and Control. This paper was recommended for publication in revised form by Associate Editor Ben M. Chen under the direction of Editor Ian R. Petersen. This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its authors.

* Corresponding author. Tel.: +1 (609) 258 0303; fax: +1 (609) 258 6109.
E-mail addresses: scardovi@princeton.edu (L. Scardovi), r.sepulchre@ulg.ac.be (R. Sepulchre).

0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2009.07.006

$A = 0$ (Moreau, 2005, 2004). The generalization includes the non-trivial examples of synchronizing harmonic oscillators and chains of integrators. In turn, these models, are applicable to distributed coordination of interconnected mobile systems. Models of vehicles often require to take into account second-order dynamics that can be feedback linearized as double integrators. Furthermore clocks synchronization problems have been reduced to synchronization of double integrator models (Carli, Chiuso, Schenato, & Zampieri, 2008).

The proposed dynamic controller structure proposed in this paper differs from the static diffusive coupling often considered in the synchronization literature, which requires more stringent assumptions on the communication graph. For instance, the results in the recent papers (Carli et al., 2008; Ren, 2008; Tuna, 2008) on synchronization of harmonic oscillators and double integrators, assume a time-invariant communication topology. The idea of designing dynamic controllers for synchronization of networked systems was recently proposed in Scardovi et al. (2007) and has been applied to stabilize planar and three-dimensional collective motion (Scardovi et al., 2008; Sepulchre et al., 2008).

The paper also summarizes several sufficient conditions for synchronization by static diffusive coupling and illustrates on simple examples that synchronization may fail under diffusive coupling when the stronger assumptions on the communication graph are not satisfied.

The paper is organized as follows. In Section 2 the notation used throughout the paper is summarized, some preliminary results are reviewed and the synchronization problem is introduced and defined. In Section 3 the main result is presented. In Section 4 we derive the discrete-time counterpart of the main result. Finally, in Section 5, two-dimensional examples are reported to illustrate the role of the proposed dynamic controller in situations where static diffusive coupling fails to achieve synchronization.

2. Preliminaries

Throughout the paper we will use the following notation. Given N vectors $x_1, x_2, \dots, x_N$ we indicate with x the stacking of the vectors, i.e. $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_N^{\mathrm{T}}]^{\mathrm{T}}$. We denote with I_N the N-dimensional diagonal matrix and we define $1_N \triangleq [1, 1, \dots, 1]^T \in \mathbb{R}^N$. Given two matrices A and B we denote their Kronecker product with $A \otimes B$. For notational convenience, we use the convention $\tilde{A}_N = I_N \otimes A$ and $\hat{A}_N = A \otimes I_N$. For a comprehensive list of properties of the Kronecker product the reader is referred to (Horn & Johnson, 1994).

2.1. Communication graphs

Given a set of interconnected systems the communication topology is encoded through a *communication graph*. The convention is that system j receives information from system i iff there is a directed link from node j to node i in the communication graph. Let $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A_d(t))$ be a time-varying weighted digraph (directed graph) where $\mathcal{V} = \{v_1, \dots, v_N\}$ is the set of nodes, $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $A_d(t)$ is a weighted adjacency matrix with nonnegative elements $a_{kj}(t)$. In the following we assume that $A_d(t)$ is piecewise continuous and bounded and $a_{kj}(t) \in \{0\} \cup [\eta, \gamma], \forall k, j,$ for some finite scalars $0 < \eta \le \gamma$ and for all $t \ge 0$. Furthermore $\{v_k, v_j\} \in \mathcal{E}(t)$ if and only if $a_{kj}(t) \ge \eta$. The set of neighbors of node v_k at time t is denoted by $\mathcal{N}_k(t) \triangleq \{v_j \in \mathcal{V} : a_{kj}(t) \ge \eta\}$. A path is a sequence of vertices such that for each of its vertices v_k the next vertex in the sequence is a neighbor of v_k. Assume that there are no self-cycles i.e. $a_{kk}(t) = 0,\ k = 1, 2, \dots, N$, and for any t.

The Laplacian matrix $L(t)$ associated to the graph $\mathcal{G}(t)$ is defined as

$$l_{kj}(t) = \begin{cases} \sum_{i=1}^{N} a_{ki}(t), & j = k \\ -a_{kj}(t), & j \neq k. \end{cases}$$

The in-degree (respectively out-degree) of node v_k is defined as $d_k^{\mathrm{in}} = \sum_{j=1}^{N} a_{kj}$ (respectively $d_k^{\mathrm{out}} = \sum_{j=1}^{N} a_{jk}$). The digraph $\mathcal{G}(t)$ is said to be *balanced* at time t if the in-degree and the out-degree of each node are equal, that is,

$$\sum_{j=1}^{N} a_{kj} = \sum_{j=1}^{N} a_{jk}, \quad k = 1, \dots, N.$$

Balanced graphs have the property that the symmetric part of their Laplacian matrix is positive semidefinite: $L + L^{\mathrm{T}} \ge 0$ (Willems, 1976). We recall some definitions that characterize the concept of connectivity for time-varying graphs.

Definition 1. The digraph $\mathcal{G}(t)$ is *connected* at time t if there exists a node v_k such that all the other nodes of the graph are connected to v_k via a path that follows the direction of the edges of the digraph.

Definition 2. Consider a graph $\mathcal{G}(t)$. A node v_k is said to be connected to node v_j ($v_j \neq v_k$) in the interval $I = [t_a, t_b]$ if there is a path from v_k to v_j which respects the orientation of the edges for the directed graph $(\mathcal{V}, \cup_{t\in I} \mathcal{E}(t), \int_I A_d(\tau)\mathrm{d}\tau)$.

Definition 3. $\mathcal{G}(t)$ is said to be uniformly connected if there exists a time horizon $T > 0$ and an index k such that for all t all the nodes v_j ($j \neq k$) are connected to node v_k across $[t, t + T]$.

2.2. Convergence of consensus algorithms

Consider N agents exchanging information about their state vector x_k, $k = 1, \dots, N$, according to a communication graph $\mathcal{G}(t)$. A classical consensus protocol in continuous-time is

$$\dot{x}_k = \sum_{j=1}^{N} a_{kj}(t)(x_j - x_k), \quad k = 1, \dots, N. \tag{1}$$

In discrete-time the analogous dynamics write

$$x_k(t+1) = x_k(t) - \epsilon_k \sum_{j=1}^{N} l_{kj}(t) x_j(t), \quad k = 1, \dots, N, \tag{2}$$

where $\epsilon_k \in (0, 1/d_k^{\mathrm{in}})$. Using the Laplacian matrix, (1) and (2) can be equivalently expressed as

$$\dot{x} = -\hat{L}_n(t)\, x \tag{3}$$

and

$$x(t+1) = \left(I_{nN} - \hat{\epsilon}\, \hat{L}_n(t)\right) x(t), \tag{4}$$

where $\hat{\epsilon} = \epsilon \otimes I_n$ and $\epsilon = \mathrm{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_N)$, and $\hat{L}_n(t) = L(t) \otimes I_n$.

Algorithms (3) and (4) have been widely studied in the literature and asymptotic convergence to a consensus value holds under mild assumptions on the communication topology. The following theorem summarizes the main results in Moreau (2005) and Moreau (2004).

Theorem 1. *Let x_k, $k = 1, 2, \dots, N$, belong to a finite-dimensional Euclidean space W. Let $\mathcal{G}(t)$ be a uniformly connected digraph and $L(t)$ the corresponding Laplacian matrix bounded and piecewise continuous in time. Then the equilibrium sets of consensus states of (3) and (4) are uniformly exponentially stable. Furthermore the solutions of (1) and (2) asymptotically converge to a consensus value β for some $\beta \in W$.* □

2.3. Problem statement

Consider N identical linear dynamical systems

$$\dot{x}_k = Ax_k + Bu_k, \tag{5a}$$

$$y_k = Cx_k, \tag{5b}$$

for $k = 1, 2, \ldots, N$, where $x_k \in \mathbb{R}^n$ is the state of the system, $u_k \in \mathbb{R}^m$ is the control vector, and $y_k \in \mathbb{R}^p$ is the output.

According to the graph-theoretic interpretation of the interconnected system described in Section 2.1, two systems are coupled at time t if there exists an edge connecting them in the associated (time-varying) communication graph $\mathcal{G}(t)$ at time t. We assume that the systems, when connected, can exchange relative information only. Thanks to this assumption the network does not need an external common reference nor a leader to synchronize. The coupling is taken into account through the inputs u_k, $k = 1, 2, \ldots, N$. We will call a control law *dynamic* if it depends on an internal (controller) state, otherwise it is called *static*. The goal of the (coupling) control law is to achieve (asymptotically) state synchronization i.e., to guarantee that $x_k - x_j \to 0$, as $t \to \infty$ for every $k, j = 1, 2, \ldots, N$.

Synchronization Problem: Given N identical linear systems described by (5) and given a uniformly connected communication graph $\mathcal{G}(t)$, find a (distributed) control law, that depends on relative information only, such that the solutions of the closed-loop system asymptotically synchronize. □

It is worth noting that in the particular case $A = 0$ and $B = C = I_n$, the problem is equivalent to the consensus problem described in Section 2.2 and the solution is well known (see Theorem 1): uniform connectivity of the graph is a sufficient condition for synchronization (or consensus) under the control law (1) or (2). The present paper addresses the generalization for linear time-invariant systems.

3. Main result

Before stating the main result we introduce a preliminary Lemma that is a direct extension of Theorem 1 when, in (5), C and B are nonsingular (square) matrices.

Lemma 1. *Consider the linear systems (5). Let B and C be $n \times n$ nonsingular matrices and assume that all the eigenvalues of A belong to the imaginary axis. Assume that the communication graph $\mathcal{G}(t)$ is uniformly connected and the corresponding Laplacian matrix $L(t)$ piecewise continuous and bounded. Then the control law*

$$u_k = B^{-1}C^{-1}\sum_{j=1}^{N} a_{kj}(t)(y_j - y_k), \quad k = 1, 2, \ldots, N, \tag{6}$$

uniformly exponentially synchronizes all the solutions of (5) to a solution of the system $\dot{x}_0 = Ax_0$. □

Proof. Control law (6) applied to system (5) yields the closed-loop system

$$\dot{x}_k = Ax_k + \sum_{j=1}^{N} a_{kj}(t)(x_j - x_k), \quad k = 1, 2, \ldots, N.$$

The change of variable

$$z_k = \mathrm{e}^{-A(t-t_0)}x_k, \quad k = 1, 2, \ldots, N$$

leads to

$$\begin{aligned}\dot{z}_k &= -A\mathrm{e}^{-A(t-t_0)}x_k + \mathrm{e}^{-A(t-t_0)}Ax_k \\ &\quad + \mathrm{e}^{-A(t-t_0)}\sum_{j=1}^{N} a_{kj}(t)(x_j - x_k) \\ &= \sum_{j=1}^{N} a_{kj}(t)(z_j - z_k)\end{aligned}$$

or, in a compact form,

$$\dot{z} = -\hat{L}_n(t)\, z.$$

From Theorem 1 the solutions $z_k(t)$, $k = 1, 2, \ldots, N$, exponentially converge to a common value $x_0 \in \mathbb{R}^n$ as $t \to \infty$, that is, there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that for all t_0,

$$\|z_k(t) - x_0\| \le \delta_1 \mathrm{e}^{-\delta_2(t-t_0)}\|z_k(t_0) - x_0\|, \quad \forall t > t_0.$$

In the original coordinates, this means

$$\begin{aligned}\left\|x_k(t) - \mathrm{e}^{A(t-t_0)}x_0\right\| &\le \delta_1 \mathrm{e}^{-\delta_2(t-t_0)}\left\|\mathrm{e}^{A(t-t_0)}\right\| \\ &\quad \times \|x_k(t_0) - x_0\|,\end{aligned}$$

for every $t > t_0$. Because all the eigenvalues of the matrix A lie on the imaginary axis, there exists a constant $\delta_3 > 0$ such that

$$\left\|x_k(t) - \mathrm{e}^{A(t-t_0)}x_0\right\| \le \delta_1 \mathrm{e}^{-\delta_3(t-t_0)}\|x_k(t_0) - x_0\|,$$

for every $t > t_0$, which proves that all solutions exponentially synchronize to a solution of the open-loop system. ■

Remark 1. The result is of course unchanged if A also possesses eigenvalues with a negative real part. Exponentially stable modes synchronize to zero, even in the absence of coupling. In contrast, the situation of systems with some eigenvalues with a positive real part can be addressed is a similar way but it requires that the graph connectivity is sufficiently strong to dominate the instability of the system. This is clear from the last part of the proof of Lemma 1 where the exponential synchronization in the z coordinates must dominate the divergence of the unstable modes of A.

We are now ready to present the main result of the paper. The assumption of *square* (nonsingular) matrices B and C in Lemma 1 are now weakened to stabilizability of the pair (A, B) and detectability of the pair (A, C). Then, for an arbitrary stabilizing feedback matrix K and observer matrix H, the (dynamic) control law

$$\begin{aligned}\dot{\eta}_k &= (A + BK)\,\eta_k + \sum_{j=1}^{N} a_{kj}(t)\left(\eta_j - \eta_k + \hat{x}_k - \hat{x}_j\right) \\ \dot{\hat{x}}_k &= A\hat{x}_k + Bu_k + H\left(\hat{y}_k - y_k\right) \\ u_k &= K\eta_k\end{aligned} \tag{7}$$

for $k = 1, 2, \ldots, N$, where $\hat{y}_k = C\hat{x}_k$, solves the Synchronization Problem.

Theorem 2. *Consider (5) and assume that the pair (A, B) is stabilizable, the pair (A, C) is detectable and that all the eigenvalues of A belong to the closed left-half complex plane. Assume that the communication graph is uniformly connected and the Laplacian matrix is piecewise continuous and bounded. Then for any gain matrices K and H such that $A+BK$ and $A+HC$ are Hurwitz, the solutions of (5) with the dynamic controller (7) uniformly exponentially synchronize to a solution of $\dot{x}_0 = Ax_0$.* □

Proof. Define $s_k = \hat{x}_k - \eta_k$ and $e_k = x_k - \hat{x}_k$, and rewrite the closed-loop system as

$$\dot{x} = \left(\tilde{A}_N + \tilde{B}_N\tilde{K}_N\right)x - \tilde{B}_N\tilde{K}_N\,(e + s) \tag{8}$$

$$\dot{s} = \tilde{A}_N s - \hat{L}_n(t)s + \tilde{H}_N\tilde{C}_N e \tag{9}$$

$$\dot{e} = \left(\tilde{A}_N + \tilde{H}_N\tilde{C}_N\right)e, \tag{10}$$

where $\tilde{C}_N = I_N \otimes C$. This system is the cascade of the system

$$\dot{x} = \left(\tilde{A}_N + \tilde{B}_N\tilde{K}_N\right)x - \tilde{B}_N\tilde{K}_N\,(e + s) \tag{11}$$

with the exponentially stable estimation error dynamics (10) and the (consensus) dynamics (9). Subsystem (9) obeys to the dynamics analyzed in Lemma 1 with an extra input that vanishes exponentially. The assumptions of Lemma 1 are satisfied for the subsystem (9) with zero input. Therefore, when $e = 0$, (9) has a uniformly exponentially stable invariant subspace that is the set where s_k, $k = 1, 2, \ldots, N$, are synchronized. Then it is possible to choose a linear coordinate change to decouple the in-manifold dynamics and the off-manifold dynamics. The off-manifold dynamics have a uniformly exponentially stable isolated equilibrium, forced by an input that vanishes exponentially. From Corollary 1 in Loria, Panteley, Popovic, and Teel (2005) we conclude that s_k, $k = 1, 2, \ldots, N$ uniformly asymptotically synchronize to a solution of $\dot{s}_0 = As_0$. Subsystem (9) can be rewritten as

$$\dot{\eta} = \left(\tilde{A}_N + \tilde{B}_N\tilde{K}_N\right)\eta + \hat{L}_n(t)s. \tag{12}$$

Since the pair (A, B) is stabilizable and K is chosen to be a stabilizing matrix, the subsystem (12) is an exponentially stable system driven by an input $\hat{L}_n(t)s(t)$ that asymptotically converges to zero. As a consequence, its solution $\eta(t)$ exponentially converges to zero as well. Since $x_k = s_k + e_k + \eta_k$ we conclude that the solutions of (5) with the dynamic controller (7) exponentially synchronize to a solution of $\dot{x}_0 = Ax_0$. ■

Some comments on Theorem 2 are now in order. The dynamic control law (7) requires that the systems exchange relative information only, i.e., the controller state differences and the state estimate differences. This fact implies that an absolute reference frame is not required. In the particular case of state coupling, i.e. when $C = I_n$, the Luenberger observer in (7) is not required and the dynamic control law simplifies to

$$\begin{aligned}\dot{\eta}_k &= (A + BK)\,\eta_k + \sum_{j=1}^{N} a_{kj}(t)\left(\eta_j - \eta_k + x_k - x_j\right),\\ u_k &= K\eta_k,\end{aligned} \tag{13}$$

for $k = 1, 2, \ldots, N$.

4. Discussion and extensions

For the sake of completeness, in this section we extend the obtained results to address synchronization of discrete-time linear systems. Then we briefly discuss the existing results (providing also a simple extension) about synchronization under (static) diffusive output coupling.

4.1. Discrete-time systems

Consider the discrete-time linear systems

$$x_k(t+1) = Ax_k(t) + Bu_k(t), \tag{14a}$$

$$y_k(t) = Cx_k(t), \tag{14b}$$

for $k = 1, 2, \ldots, N$, and $t = 1, 2, \ldots$.

Now consider system (14) with the dynamic control law (the discrete-time version of (7))

$$\begin{aligned}\eta_k(t+1) &= (A + BK)\,\eta_k(t) + \epsilon_k A\sum_{j=1}^{N} l_{kj}(t)\left(\hat{x}_j(t) - \eta_j(t)\right),\\ \hat{x}_k(t+1) &= A\hat{x}_k(t) + H\left(y_k(t) - \hat{y}_k(t)\right),\end{aligned} \tag{15}$$

for $k = 1, 2, \ldots, N$, where $\hat{y}_k(t) = C\hat{x}_k(t)$.

Theorem 2 can be easily adapted to the case of discrete-time linear systems. The proof is based on an analogous version of Lemma 1 for discrete-time systems and on an adaptation of the proof of Theorem 2 (see the paper Scardovi and Sepulchre (2008), available online, for more details).

Theorem 3. *Consider (14) and assume that the pair (A, B) is stabilizable, the pair (A, C) is detectable and that all the eigenvalues of A belong to the closed unitary disk in the complex plane. Assume that the communication graph $\mathcal{G}(t)$ is uniformly connected and the Laplacian matrix is piecewise continuous and bounded. Then for any gain matrices K and H such that $A+BK$ and $A+HC$ are Schur matrices, the solutions of (14) with the dynamic controller (15) uniformly exponentially synchronize to a solution of $x_0(t+1) = Ax_0(t)$.* □

Remark 2. The results presented in the present paper can be extended to (time-varying) periodic linear systems. Periodic linear systems, naturally arise in a number of contexts in engineering, physics, and biology (Bittanti, 2001). Furthermore, periodic linear systems emerge from the linearization of nonlinear oscillators along the periodic solutions. We refer the interested reader to the paper (Scardovi & Sepulchre, 2008) (available online) for a treatment of this subject.

4.2. Dynamic coupling versus static coupling

Theorem 2 and Theorem 3 provide a fairly general solution to the Synchronization Problem through the construction of a dynamic output feedback controller. For the sake of comparison, we provide a set of sufficient conditions to prove synchronization under a simple static output feedback (diffusive) interconnection. These sufficient conditions require stronger assumptions on the interconnection and assume a passivity property for the system (A, B, C), that is, the existence of a symmetric positive definite matrix $P > 0$ that verifies

$$PA + A^{\mathrm{T}}P \le 0, \qquad B^{\mathrm{T}}P = C. \tag{16}$$

Passivity conditions have been considered previously in Arcak (2007) (where it is assumed that the communication topology is bidirectional and strongly connected) and in Stan and Sepulchre (2007) (where synchronization is studied for a class of (nonlinear) oscillators assuming that the communication topology is time-invariant and balanced). Assumptions A1 and A2 below lead to a time-varying extension of the results in Stan and Sepulchre (2007) and Arcak (2007) in the special case of linear systems.

Theorem 4. *Consider system (5) with the static output feedback control laws*

$$u_k = \sum_{j=1}^{N} a_{kj}(t)(y_j - y_k), \quad k = 1, 2, \ldots, N.$$

Let the graph Laplacian matrix $L(t)$ be piecewise continuous and bounded. Then exponential synchronization to a solution of $\dot{x}_0 = Ax_0$ is achieved under either one of the following assumptions:

A1. *The system (A, B, C) is passive and observable, the communication graph is connected and balanced at each time;*

A2. *The system (A, B, C) is passive and observable, the communication graph is symmetric, i.e. the Laplacian matrix can be factorized as $L = DD^{\mathrm{T}}(t)$, and the pair $(\tilde{A}_N, \hat{D}_p^{\mathrm{T}}(t)\tilde{C}_N)$, is uniformly observable, where $\hat{D}_p = D \otimes I_p$.* □

Due to the space constraints, we refer to the paper (Scardovi & Sepulchre, 2008) available online for the proof.

5. Examples

The conditions of Theorem 4 are only *sufficient* conditions for exponential synchronization under diffusive coupling. We provide two simple examples to illustrate that these conditions are not far from being necessary when considering time-varying and directed graphs and that the *internal model* of the dynamic controller (7) plays an important role in such situations.

Example 1. (Synchronization of Harmonic Oscillators)
Consider a group of N harmonic oscillators

$$\begin{aligned}\dot{x}_{1k} &= x_{2k},\\ \dot{x}_{2k} &= -x_{1k} + u_k,\\ y_k &= x_{2k},\end{aligned} \qquad (17)$$

for $k = 1, 2, \ldots, N$, which corresponds to system (5) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = (0\ 1)\,.$$

The assumptions of Theorem 4 are satisfied: A is Lyapunov stable, (A, B) is stabilizable and (A, C) is detectable. Choosing the stabilizing gain $K = (0\ -1)$ and the observer matrix $H = (0\ -1)^{\mathrm{T}}$, the dynamic control law (7) particularizes to

$$\begin{aligned}u_k &= -\eta_{2k},\\ \dot{\eta}_{1k} &= \eta_{2k} + \sum_{j=1}^{N} a_{kj}(t)(\eta_{1j} - \eta_{1k} + \hat{x}_{1k} - \hat{x}_{1j}),\\ \dot{\eta}_{2k} &= -\eta_{1k} - \eta_{2k} + \sum_{j=1}^{N} a_{kj}(t)(\eta_{2j} - \eta_{2k} + \hat{x}_{2k} - \hat{x}_{2j}),\\ \dot{\hat{x}}_{1k} &= \hat{x}_{2k},\\ \dot{\hat{x}}_{2k} &= -\hat{x}_{1k} - \eta_{2k} + y_k - \hat{y}_k,\end{aligned} \qquad (18)$$

for $k = 1, 2, \ldots, N$, where $\hat{y}_k = \hat{x}_{2k}$ and $u_k = -\eta_{2k}$. Theorem 4 ensures exponential synchronization of the oscillators to a solution of the harmonic oscillator if the graph is uniformly connected. Fig. 2 illustrates the simulation of a group of 4 oscillators coupled according to the time-varying communication topology shown in Fig. 1 (the period T is set to 7 s). The dynamic control law (18) ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection

$$u_k = \sum_{j=1}^{N} a_{kj}(t)(x_{2j} - x_{2k}). \qquad (19)$$

The system (A, B, C) is nevertheless passive, meaning that stronger assumptions on the communication graph would ensure synchronization with the diffusive coupling (19). We mention the recent result (Tuna, 2008) that proves (in discrete-time) synchronization of harmonic oscillators with diffusive coupling under the assumption that the graph is time-invariant and connected.

Remark 3. The harmonic oscillators considered in this example are passive. It is worth noting that Theorem 2 does not require passivity and analogous results are obtained, e.g., when $C = (1\ 0)$ picking $H = (0\ -1)^{\mathrm{T}}$ (i.e. when the harmonic oscillators are not passive).

The following example illustrates an analog scenario with unstable dynamics.

Example 2. (Synchronization of Double Integrators) Consider a group of N double integrators

$$\begin{aligned}\dot{x}_{1k} &= x_{2k},\\ \dot{x}_{2k} &= u_k,\\ y_k &= x_{1k} + x_{2k},\end{aligned} \qquad (20)$$

for $k = 1, 2, \ldots, N$, which corresponds to system (5) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (1\ 1)\,.$$

The assumptions of Theorem 4 are satisfied: the two eigenvalues of A are zero and (A, B) is stabilizable. Choosing the stabilizing gain $K = (-1\ -1)$ and the observer matrix $H = (0\ -1)^{\mathrm{T}}$, the dynamic control law (13) yields closed-loop system

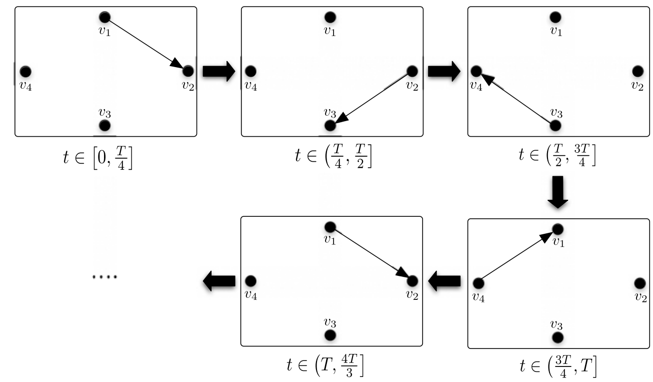

Fig. 1. The time-varying communication topology (in the particular case of $N = 4$) used in Examples 1 and 2.

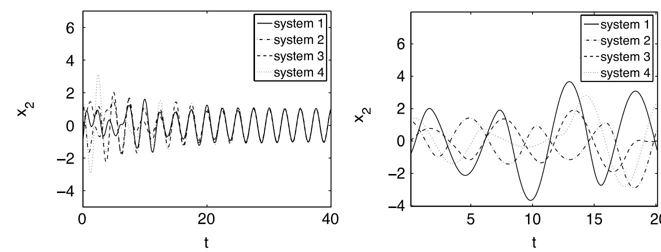

Fig. 2. Evolution of the first component of the solutions for the closed-loop harmonic oscillators resulting from system (17) with the dynamic control law (18) (to the left) and the static control law (19) (to the right). The dynamic control ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection.

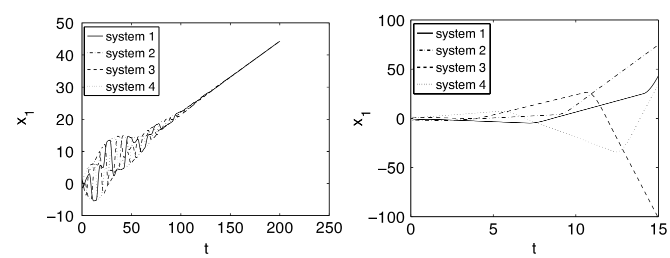

Fig. 3. Evolution of the first component of the solutions for the closed-loop double integrators resulting from system (20) with the dynamic control law (21) (to the left) and the static control law (22) (to the right). The dynamic control ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection.

$$\begin{aligned}u_k &= -\eta_{1k} - \eta_{2k},\\ \dot{\eta}_{1k} &= \eta_{2k} + \sum_{j=1}^{N} a_{kj}(t)(\eta_{1j} - \eta_{1k} + x_{1k} - x_{1j}),\\ \dot{\eta}_{2k} &= -\eta_{1k} - \eta_{2k} + \sum_{j=1}^{N} a_{kj}(t)(\eta_{2j} - \eta_{2k} + x_{2k} - x_{2j}),\\ \dot{\hat{x}}_{1k} &= \hat{x}_{2k},\\ \dot{\hat{x}}_{2k} &= -\eta_{1k} - \eta_{2k} + y_k - \hat{y}_k,\end{aligned} \qquad (21)$$

for $k = 1, 2, \ldots, N$, where $\hat{y}_k = \hat{x}_{1k} + \hat{x}_{2k}$. Theorem 4 ensures exponential synchronization to a solution of the double integrator if the graph is uniformly connected. Fig. 3 illustrates the simulation of a group of 4 double integrators coupled according to the time-varying communication topology shown in Fig. 1 (the period T is set to 7 s). The dynamic control (21) ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection

$$u_k = \sum_{j=1}^{N} a_{kj}(t)(y_j - y_k), \quad y_k = x_{1k} + x_{2k}. \tag{22}$$

The matrix $A - \alpha BC$ is nevertheless stable for every $\alpha > 0$, suggesting that stronger assumptions on the communication graph would ensure synchronization.

6. Conclusions

In this paper the problem of synchronizing a network of identical linear systems described by the state-space model (A, B, C) has been addressed. A dynamic controller ensuring exponential convergence of the solutions to a synchronized solution of the decoupled systems is provided assuming that (i) A has no exponentially unstable modes, (ii) (A, B) is stabilizable and (A, C) is detectable, and (iii) the communication graph is uniformly connected. Stronger conditions are shown to be sufficient (and, to some extent, also necessary) to ensure synchronization with the often considered static diffusive output coupling.

References

Arcak, M. (2007). Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, *52*(18), 1380–1390.

Bittanti, S., & Collaneri, P. (Eds.). (2001). *Periodic control systems 2001*. Italy: Cernobbio-Como.

Carli, R., Chiuso, A., Schenato, L., & Zampieri, S. (2008). A pi consensus controller for networked clocks synchronization. In *17th IFAC world congress*.

Hale, J. K. (1996). Diffusive coupling, dissipation, and synchronization. *Journal of Dynamics and Differential Equations*, *9*(1), 1–52.

Horn, R. A., & Johnson, C. R. (1994). *Topics in matrix analysis*. Cambridge University Press.

Jadbabaie, A., Lin, J., & Morse, S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, *48*, 988–1001.

Loria, A., Panteley, E., Popovic, D., & Teel, A. (2005). A nested matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems. *IEEE Transactions on Automatic Control*, *50*(2), 183-–198.

Moreau, L. (2005). Stability of multi-agent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, *50*(2), 169–182.

Moreau, L. (2004). Stability of continuous-time distributed consensus algorithms. In *Proceedings of the 43rd IEEE conference on decision and control*, (pp. 3998–4003).

Nair, S., & Leonard, N. (2008). Stable synchronization of mechanical system networks. *SIAM Journal on Control and Optimization*, *47*(2), 661–683.

Olfati-Saber, R., & Murray, R. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.

Pham, Q. C., & Slotine, J.-J. (2007). Stable concurrent synchronization in dynamic system networks. *Neural Networks*, *20*(1), 62–77.

Pogromsky, A. (1998). Passivity based design of synchronizing systems. *International Journal of Bifurcation and Chaos*, *8*(2), 295–319.

Ren, W. (2008). On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control*, *53*(6), 1503–1509.

Sarlette, A., Sepulchre, R., & Leonard, N. (2007). Autonomous rigid body attitude synchronization. In *Proceedings of the 46th IEEE conference on decision and control*, (pp. 2566–2571.

Scardovi, L., Leonard, N., & Sepulchre, R. (2008). Stabilization of collective motion in three-dimensions. *Communications in Information and Systems*, *8*(3), 473–500 (Special issue dedicated to the 70th birthday of Roger W. Brockett).

Scardovi, L., Sarlette, A., & Sepulchre, R. (2007). Synchronization and balancing on the N-torus. *Systems and Control Letters*, *56*(5), 335–341.

Scardovi, L., & Sepulchre, R. (2008). Synchronization in networks of identical linear systems. arXiv:0805.3456v1 [Online]. Available: http://arxiv.org/abs/0805.3456.

Sepulchre, R., Paley, D., & Leonard, N. (2008). Stabilization of planar collective motion with limited communication. *IEEE Transactions on Automatic Control*, *53*(3), 706–719.

Stan, G. B., & Sepulchre, R. (2007). Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, *52*(2), 256–270.

Tuna, E. S. (2008). Synchronizing linear systems via partial-state coupling. *Automatica*, *44*, 2179–2184.

Willems, J. C. (1976). Lyapunov functions for diagonally dominant systems. *Automatica. A Journal IFAC*, *12*(5), 519–523.

Luca Scardovi received his engineering degree and Ph.D. degree in Electronic and Computer Engineering from the University of Genoa, Italy, in 2001 and 2005 respectively. In 2005 he was Adjunct Professor at the University of Salento, Italy. He held research associate positions at the Department of Electrical Engineering and Computer Science at the University of Liège, Belgium (2005–2007) and at the Department of Mechanical and Aerospace Engineering at Princeton University, USA (2007–2009). He is currently an Assistant Professor in the Department of Electrical Engineering and Information Technology at the Technische Universität München (TUM), Germany. His research interests focus on dynamical systems with special emphasis in analysis and control of complex networked systems.

Rodolphe Sepulchre is Professor in the Department Electrical Engineering and Computer Science, Université de Liège. He received his engineering degree and Ph.D. degree in applied mathematics from the University of Louvain, Belgium, in 1990 and 1994 respectively. He held research or teaching positions at the University of California, Santa Barbara (1994–1996), Princeton (2002–2003), and Louvain (2001–2009). His research interests include nonlinear control systems, dynamical systems, and optimization on manifolds. He is the recipient of the 2008 IEEE Antonio Ruberti prize.