

# Invariance Controlled Balance of Legged Robots

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**Abstract**—A balance control approach for legged robots is presented that is based on Zero Moment Point manipulation. Therefore the invariance control method is used, where an output of the system is kept in an admissible region by switching between two controllers, the nominal controller and the corrective controller. The nominal controller is designed to achieve the main control objective whereas the corrective controller is used to keep the system in the admissible region whenever a constraint violation is about to occur. The corrective control signal is chosen as close as possible to the nominal control signal to enable the return to the nominal task after correction is not necessary any more. The control system for balance control comprises the mechanical robot model and a simplified model of the motor dynamics. The method is demonstrated in simulation experiments of a single joint robot and a planar eight degree of freedom legged robot.

## I. INTRODUCTION

One goal in legged robot control is to equip the robots with controllers such that they maintain balance or in other words that they avoid underactuated contact where rotation around foot edges occurs. The control of the Zero Moment Point (ZMP) is a common concept for this task. The ZMP, introduced by Vukobratović [1], is the point on ground level, where a contact force has to act to balance the robot. Thus, if the ZMP is in the foot-ground contact area or inside the convex hull of the contact areas of multiple contacting feet, tilting around foot edges will not occur. Therefore, a ZMP control algorithm is necessary that keeps the ZMP in the convex hull of the foot margins. Many research groups work on approaches for balance control of legged robots [2]–[7]. Challenges in ZMP control are the non-minimum phase character of the control system and the interaction of the ZMP control and the nominal control task.

Often, a rather intuitive control law is used for balance control in experimental studies. The reference trajectory for the ankle joint is altered proportionally to the deviation of the measured ZMP from the desired ZMP [2]. This control scheme has a severe theoretical drawback: The control torque is set depending on the actual ZMP deviation, while the ZMP directly depends on the torque, which results in an algebraic loop. This approach is not applicable in simulations without solving the algebraic loop or introducing artificial time delays. In experiments the method is nevertheless successful, also because motor dynamics and flexibilities in the links introduce additional dynamics that negate the direct dependence of the ZMP on the inputs. But for this

setting analytical stability proofs are difficult to obtain. Theoretically consistent formulations explicitly consider the algebraic relationship of the ZMP and the motor torques [3], [4] and calculate the appropriate motor torques that either correct the ZMP or track the desired trajectory. In the control method for the humanoid Johnnie [4] the control of the center of gravity is replaced by control of the contact forces whenever a ZMP violation is about to occur. Kondak *et al.* [3] derive a ZMP-plane that defines valid accelerations. Then the actual acceleration is projected on this plane to avoid violation of the ZMP constraint equation. In contrast to experimental approaches these two approaches are model-based and allow exact ZMP tracking.

The approach to ZMP control presented in this paper is similar to the approaches in [3], [4] that also use model-based ZMP tracking by manipulation of accelerations. The ZMP control problem will be addressed in the control theoretic framework of invariance control [8], [9]. This control method considers control affine systems with constraints defined by zero sublevel sets of an output function. A constraint admissible state space region is calculated from the constraints, which can be kept invariant by switching of the control between a nominal controller and a corrective controller. The nominal controller is designed to achieve the main control objective while the constraints are not violated. Control is switched to the corrective controller when a constraint violation is imminent. In this paper, the invariance control method is extended for multi-input systems and applied to the ZMP control problem. When the constraint imposed by the ZMP balance condition is formulated as a system output, this output will directly depend on the motor torques. By consideration of the motor dynamics, the control inputs do no longer directly modify the ZMP, but its first time derivative. This avoids the algebraic loop and allows for a consistent control theoretic formulation of the ZMP control problem.

In Sec. II the problem of balance control for legged robots is outlined. Then in Sec. III the invariance control method is presented to prepare its application for balance control in Sec. IV. The paper is resumed with a discussion in Sec. VI and a conclusion in Sec. VII.

## II. PROBLEM STATEMENT

Legged robots usually consist of rigid links that are connected by rotatoric joints. The considered robots are assumed to have rigid flat feet such that they are fully actuated as long

as the foot has flat contact. One example robot is displayed in Fig. 1. It is made up of two legs, an upper body including the head, and an arm. The control task is to avoid underactuation that occurs when the robot rotates around foot edges after the ground reaction force did not succeed to compensate the rotational moment. In what follows, only planar robot constructions are considered.

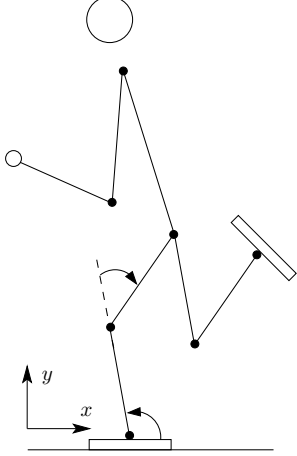


Fig. 1. Legged robot.

To obtain a dynamical model of a legged robot, a set of minimal coordinates is chosen, which are the joint angles  $\mathbf{q} = (q_1, \dots, q_n)^T \in \mathbb{R}^n$ . The joints are actuated by torques that motors apply in the joints, denoted by  $\boldsymbol{\tau} \in \mathbb{R}^n$ . The following general expression describes the dynamics of a legged robot, and can e.g. be derived from Euler-Lagrange equations [10]:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}) = \boldsymbol{\tau} \quad (1)$$

Here  $\mathbf{M}(\mathbf{q})$  is the symmetric inertia matrix and  $\mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$  is the vector of coriolis, centrifugal, and gravitational forces and torques.

Further we assume a simple first-order model for the motor dynamics that relates acting torques  $\boldsymbol{\tau}$  and input currents  $\mathbf{v} \in \mathbb{R}^n$  of the motor:

$$\dot{\boldsymbol{\tau}} + \mathbf{K}\boldsymbol{\tau} = \mathbf{v} \quad (2)$$

The constant diagonal matrix  $\mathbf{K}$  summarizes the time constants of the motor dynamics. Hence, the robot dynamics (1), (2) are a control system with state vector

$$\begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \\ \boldsymbol{\tau} \end{pmatrix} \in \mathbb{R}^{3n}$$

and control input  $\mathbf{v} \in \mathbb{R}^n$ .

The Zero Moment Point (ZMP) is the point on ground level where a vertical force has to act to balance the system [1]. If the ZMP is in the supporting area, which is the region covered by the foot, the ground reaction force balances the system. If the ZMP leaves the supporting area, the robot cannot be balanced by the ground reaction force any more and starts tipping over the respective foot edge.

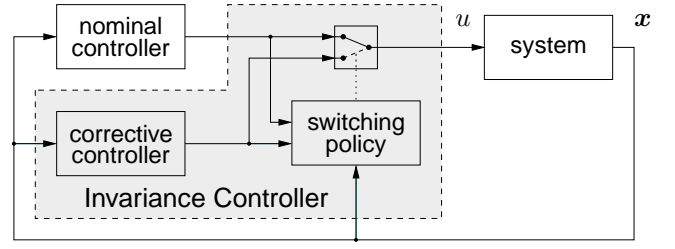


Fig. 2. Structure of the invariance control approach.

The ZMP can be derived from the equilibrium equation of horizontal moments

$$r_{\text{zmp}} = \frac{T_z}{R_y}, \quad (3)$$

where  $T_z$  is the vertical contact moment and  $R_y$  is the horizontal contact force. The contact forces and moments are calculated from the posture  $\mathbf{q}$ , the velocities  $\dot{\mathbf{q}}$ , and the torques  $\boldsymbol{\tau}$ . Thus, the ZMP is a function of the system states and can be considered as control output of the control system (1), (2).

$$y = r_{\text{zmp}}(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\tau}) \quad (4)$$

The condition for balance is that the ZMP remains in the contact area, which is, for the planar case, delimited by a lower bound  $y_l$  and an upper bound  $y_u$ .

$$y_l < r_{\text{zmp}} < y_u \quad (5)$$

This balance condition is regarded as inequality constraint for the output of the control system.

The invariance control strategy relies on information how the system output and the system input are related. The relative degree  $r$  of an output function is the order of the derivative of the output function where a direct dependence on the input occurs for the first time. For the ZMP the relative degree is  $r = 1$ . This becomes apparent, when the first derivative is calculated:

$$\begin{aligned} \frac{d}{dt} r_{\text{zmp}}(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\tau}) &= \frac{\partial r_{\text{zmp}}}{\partial \mathbf{q}} \dot{\mathbf{q}} + \frac{\partial r_{\text{zmp}}}{\partial \dot{\mathbf{q}}} \ddot{\mathbf{q}} + \frac{\partial r_{\text{zmp}}}{\partial \boldsymbol{\tau}} \dot{\boldsymbol{\tau}} \\ &= \frac{\partial r_{\text{zmp}}}{\partial \mathbf{q}} \dot{\mathbf{q}} + \frac{\partial r_{\text{zmp}}}{\partial \dot{\mathbf{q}}} \mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{n}) + \frac{\partial r_{\text{zmp}}}{\partial \boldsymbol{\tau}} (\mathbf{v} - \mathbf{K}\boldsymbol{\tau}) \end{aligned}$$

Here (1) and (2) are used to substitute  $\ddot{\mathbf{q}}$  and  $\dot{\boldsymbol{\tau}}$ . Thus,  $\dot{r}_{\text{zmp}}$  and  $\mathbf{v}$  are algebraically related in nonsingular configurations.

### III. INVARIANCE CONTROL

In the following, the invariance control approach will be introduced, highlighting the results necessary to address the ZMP control problem.

#### A. Concept

A state space region is called positive (negative) invariant with respect to a dynamical system, if trajectories originating from the region remain therein with increasing (decreasing) time. All following statements concerning invariance properties will implicitly refer to positive invariance.

The main idea of the invariance control approach is to make a subset of state space invariant by means of a switching controller, the structure of which is illustrated in Fig.2. With appropriate choice of this admissible state space subset, resulting trajectories will never cause violation of output constraints. In normal operation, the plant is controlled by a nominal controller that achieves the main control goals. When the system trajectory hits the boundary of the admissible set, the control is switched to the corrective controller, which guarantees that the system states remain inside the admissible set. In this mode, the system moves along the boundary of the admissible set. Control is switched back to the nominal controller, as soon as the nominal control no longer results in constraint violation.

### B. Definitions

The invariance control methodology considers nonlinear control affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \quad \mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m \quad (6)$$

with initial condition  $\mathbf{x}_0$  and input variable  $\mathbf{u}$ .

A general class of state constraints is defined by the zero sublevel set of a possibly fictive output

$$y = h(\mathbf{x}) \leq 0 \quad y \in \mathbb{R}. \quad (7)$$

The functions  $\mathbf{f}$ ,  $\mathbf{G}$ , and  $h$  are assumed to be sufficiently smooth to allow for the calculation of time derivatives of the output  $y$  up to the relative degree.

A point in state-space  $\mathbf{x}$  is called constraint admissible if condition (7) is satisfied. A subset of state-space is called constraint admissible if each of its elements is constraint admissible.

To assure that the set of constraint admissible states

$$\{\mathbf{x} \mid h(\mathbf{x}) \leq 0\},$$

which is also called the maximal constraint admissible set, has the boundary

$$\{\mathbf{x} \mid h(\mathbf{x}) = 0\},$$

which separates the constraint admissible states from the non-admissible states, zero has to be a regular value of the function  $h(\mathbf{x})$ . Since  $h(\mathbf{x})$  is differentiable with respect to  $\mathbf{x}$ , zero is a regular value of  $h(\mathbf{x})$  if and only if the gradient  $\frac{\partial h}{\partial \mathbf{x}}$  is non-vanishing for all  $\mathbf{x}^*$  with  $h(\mathbf{x}^*) = 0$ :

$$\left. \frac{\partial h}{\partial \mathbf{x}} \right|_{\mathbf{x}^*} \neq \mathbf{0}^T$$

### C. Design

For the control of the ZMP, it will be sufficient to consider constraints with relative degree one, since the motor torques directly manipulate the first time derivative of the constrained variable, the ZMP.

We can therefore assume that the first time derivative of the output  $y$

$$y(\mathbf{x}) = h(\mathbf{x}) \quad (8)$$

$$\frac{d}{dt}y(\mathbf{x}) = \frac{\partial h}{\partial \mathbf{x}} [\mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u}] =: \dot{y}(\mathbf{x}, \mathbf{u}) \quad (9)$$

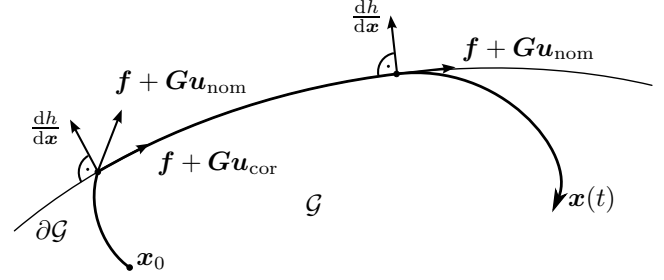


Fig. 3. Illustration of the invariance control law.

depends on the control variable  $\mathbf{u}$  and that the vector

$$\frac{\partial h}{\partial \mathbf{x}} \mathbf{G}(\mathbf{x}) \neq \mathbf{0}^T \quad (10)$$

is nonzero. This matter will be discussed in more detail in the next section, where the robotic context allows for a meaningful interpretation with respect to the balance control problem.

For the special case of constraints with relative degree one, the maximal constraint admissible set

$$\mathcal{G} = \{\mathbf{x} \mid h(\mathbf{x}) \leq 0\}$$

with boundary

$$\partial \mathcal{G} = \{\mathbf{x} \mid h(\mathbf{x}) = 0\}.$$

can be kept invariant by proper choice of the control on  $\partial \mathcal{G}$ . For higher relative degrees the corrective control will also have to act inside  $\mathcal{G}$ .

Since  $h(\mathbf{x})$  is continuously differentiable, the following more convenient invariance condition can be used, which is equivalent to the classical one that the tangent cone to  $\mathcal{G}$  in  $\mathbf{x}$  must contain the system drift vector  $\dot{\mathbf{x}}$  for all  $\mathbf{x} \in \partial \mathcal{G}$  [11]: The set  $\mathcal{G}$  is invariant if and only if the control  $\mathbf{u}$  satisfies

$$\forall \mathbf{x} \in \partial \mathcal{G} \quad \frac{\partial h}{\partial \mathbf{x}} [\mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u}] \leq 0 \quad (11)$$

for all  $\mathbf{x}$  on the boundary  $\partial \mathcal{G}$ .

To keep  $\mathcal{G}$  invariant, control is switched between nominal control and corrective control. The nominal controller is specific to the application and is used in the interior of  $\mathcal{G}$ . On the boundary  $\partial \mathcal{G}$ , control is switched to the corrective controller to satisfy the invariance condition (11) and to keep the system on the boundary.

$$\mathbf{u} = \begin{cases} \mathbf{u}_{\text{nom}} & y(\mathbf{x}) < 0 \\ \mathbf{u}_{\text{nom}} & y(\mathbf{x}) \geq 0, \dot{y}(\mathbf{x}, \mathbf{u}_{\text{nom}}) < 0 \\ \mathbf{u}_{\text{cor}} & y(\mathbf{x}) \geq 0, \dot{y}(\mathbf{x}, \mathbf{u}_{\text{nom}}) \geq 0 \end{cases} \quad (12)$$

This is illustrated in Fig.3, where the control is modified on the boundary as long as the system drift vector under nominal control points outwards of  $\mathcal{G}$ .

The corrective control signal  $\mathbf{u}_{\text{cor}}$  is derived from the equation

$$\begin{aligned} \dot{y}(\mathbf{x}, \mathbf{u}) &= 0 \\ \Leftrightarrow \underbrace{\frac{\partial h}{\partial \mathbf{x}} \mathbf{G}(\mathbf{x})}_{\mathbf{a}^T} \mathbf{u} &= - \underbrace{\frac{\partial h}{\partial \mathbf{x}} \mathbf{f}}_b \end{aligned} \quad (13)$$

It becomes visible that (11) can be satisfied by proper choice of  $\mathbf{u}$  with additional degrees of freedom as long as  $\frac{\partial h}{\partial \mathbf{x}} \mathbf{G}(\mathbf{x})$  remains non-zero, which is assured by the relative degree assumption (10).

The solution of the under-determined linear equation  $\mathbf{a}^T \mathbf{u} = b$  with minimal norm  $\|\mathbf{u}\|$  can be obtained using the pseudoinverse  $(\mathbf{a}^T)^+$  of  $\mathbf{a}^T$ :

$$\begin{aligned} \min \|\mathbf{u}\|^2 \\ \mathbf{a}^T \mathbf{u} &= b \\ \Rightarrow \mathbf{u} &= (\mathbf{a}^T)^+ b \end{aligned} \quad (14)$$

To stay as close as possible to the nominal control  $\mathbf{u}_{\text{nom}}$ , it may be better to select a control  $\mathbf{u}$  that solves (13) and minimizes the deviation from the nominal control.

$$\begin{aligned} \min \|\mathbf{u} - \mathbf{u}_{\text{nom}}\|^2 \\ \mathbf{a}^T \mathbf{u} &= b \\ \Rightarrow \mathbf{u} &= (\mathbf{a}^T)^+ b + (\mathbf{I} - (\mathbf{a}^T)^+ \mathbf{a}^T) \mathbf{u}_{\text{nom}} \end{aligned} \quad (15)$$

Here  $\mathbf{I}$  denotes the identity matrix. The validity of the solution is seen when subtracting  $\mathbf{a}^T \mathbf{u}_{\text{nom}}$  on both sides of equation (13)

$$\mathbf{a}^T (\mathbf{u} - \mathbf{u}_{\text{nom}}) = b - \mathbf{a}^T \mathbf{u}_{\text{nom}}, \quad (16)$$

because then the problem is stated in a way that again allows to find the solution by application of the pseudoinverse.

In some cases it may be desirable to minimize a weighted deviation from nominal control

$$\begin{aligned} \min \|\mathbf{W}^{-1}(\mathbf{u} - \mathbf{u}_{\text{nom}})\|^2 \\ \mathbf{a}^T \mathbf{u} &= b \\ \Rightarrow \mathbf{u} &= \mathbf{W}(\mathbf{a}^T \mathbf{W})^+ b + (\mathbf{I} - \mathbf{W}(\mathbf{a}^T \mathbf{W})^+ \mathbf{a}^T) \mathbf{u}_{\text{nom}}. \end{aligned} \quad (17)$$

Here, reformulation of equation (16) yields

$$\mathbf{a}^T \mathbf{W} [\mathbf{W}^{-1} (\mathbf{u} - \mathbf{u}_{\text{nom}})] = b - \mathbf{a}^T \mathbf{u}_{\text{nom}},$$

where the minimum norm solution (17) is obtained by pseudoinverse application again. If a diagonal weighting matrix is chosen, the  $i$ -th diagonal element  $w_{i,i}$  specifies the influence of the  $i$ -th element of  $\mathbf{u}$  with respect to the other elements.

#### IV. BALANCE CONTROL

In what follows, the invariance control method that was introduced in Sec. III is applied to balance control of legged robots. The balance control problem was outlined in Sec. II, giving reference to the underlying equations of motion and discussing necessary output constraints for the output ZMP that guarantee balance.

#### A. Control System

A first summary of the legged robot control system subject to balance control is

$$\begin{aligned} \begin{pmatrix} \ddot{\mathbf{q}} \\ \dot{\boldsymbol{\tau}} \end{pmatrix} &= \begin{pmatrix} \mathbf{M}^{-1}(\mathbf{q}) [\boldsymbol{\tau} - \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})] \\ \mathbf{v} - \mathbf{K}\boldsymbol{\tau} \end{pmatrix} \\ y &= r_{\text{zmp}}(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\tau}) \end{aligned} \quad (18)$$

Compare (1), (2) for the equations of motion and (4) for the output function. In the following a transformation by feedback-linearization is presented that will simplify the design of the nominal and the invariance control law.

For the transformation,  $\mathbf{q}$  is repeatedly differentiated, up to its relative degree.

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{v} - \mathbf{K}\boldsymbol{\tau} - \dot{\mathbf{n}}) + \frac{d\mathbf{M}^{-1}}{dt}(\boldsymbol{\tau} - \mathbf{n}) \quad (19)$$

Here (18) is used to replace  $\dot{\boldsymbol{\tau}}$ . The control system is said to be input-state linearizable because the linearization does not have an internal dynamics. Demanding  $\ddot{\mathbf{q}} = \mathbf{u}$  results in an expression  $\mathbf{v}(\mathbf{u})$  that linearizes the control system (18).

$$\mathbf{v}(\mathbf{u}) = \mathbf{M}\mathbf{u} + \frac{d\mathbf{M}}{dt} \mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{n}) + \mathbf{K}\boldsymbol{\tau} + \dot{\mathbf{n}}.$$

Here, we use the relation

$$\frac{d\mathbf{M}^{-1}}{dt} = -\mathbf{M}^{-1} \frac{d\mathbf{M}}{dt} \mathbf{M}^{-1}.$$

The transformed control system is given in the coordinates  $\mathbf{x} = (\mathbf{q}^T, \dot{\mathbf{q}}^T, \ddot{\mathbf{q}}^T)^T$ , with auxiliary input  $\mathbf{u}$ . It can be summarized as

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \\ \mathbf{0} \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}}_{\mathbf{G}(\mathbf{x})} \mathbf{u}$$

to account for the formulation in (6). The ZMP balance condition  $y_l < r_{\text{zmp}}(\mathbf{x}) < y_u$ , see (5), can be restated by two constrained output functions

$$y_1 = r_{\text{zmp}}(\mathbf{x}) - y_u \leq 0 \quad (20)$$

$$y_2 = -r_{\text{zmp}}(\mathbf{x}) + y_l \leq 0, \quad (21)$$

compare (7). Although there are two constraints, the presented invariance control method can be used because the constraints  $y_1$  and  $y_2$  are not competing. That means it will never occur that both constraints are violated at the same time.

#### B. Nominal Controller

The presented feedback linearization allows for a linear controller synthesis for the nominal control  $\mathbf{u} = \mathbf{u}_{\text{nom}}$ :

$$\mathbf{u}_{\text{nom}} = \ddot{\mathbf{q}}^d + \mathbf{K}_A(\ddot{\mathbf{q}}^d - \ddot{\mathbf{q}}) + \mathbf{K}_D(\dot{\mathbf{q}}^d - \dot{\mathbf{q}}) + \mathbf{K}_P(\mathbf{q}^d - \mathbf{q})$$

The control parameters  $\mathbf{K}_A$ ,  $\mathbf{K}_D$ , and  $\mathbf{K}_P$  are used to parameterize the linear control law. Trajectory tracking of desired trajectories  $\mathbf{q}^d$  is then realized as nominal behavior. This method is known as computed torque control in the robotics literature.

### C. Corrective Controller

The first derivative of the ZMP  $\dot{r}_{zmp}(\mathbf{x})$  depends on the input  $\mathbf{u}$ , but not the ZMP  $r_{zmp}(\mathbf{x})$  itself, as a result the system outputs  $y_1$  from (20) and  $y_2$  from (21) have relative degree one. The equation to determine the invariance control signal is thus derived from

$$\dot{y}_1 = \frac{\partial r_{zmp}}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}} + \frac{\partial r_{zmp}}{\partial \ddot{\mathbf{q}}} \ddot{\mathbf{q}} + \frac{\partial r_{zmp}}{\partial \ddot{\ddot{\mathbf{q}}}} \ddot{\ddot{\mathbf{q}}}$$

for the upper bound and analog for the lower bound. Since  $\ddot{\mathbf{q}} = \mathbf{u}$  is achieved by feedback linearization the invariance condition is linear in the control input  $\mathbf{u}_{cor}$ , compare (14).

$$\mathbf{a}^T \mathbf{u}_{cor} = b \quad (22)$$

$$\text{with } \mathbf{a}^T = \frac{\partial r_{zmp}}{\partial \ddot{\mathbf{q}}} \text{ and } b = -\frac{\partial r_{zmp}}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}} - \frac{\partial r_{zmp}}{\partial \ddot{\mathbf{q}}} \ddot{\mathbf{q}}$$

Equation (22) can be solved for  $\mathbf{u}_{cor}$  but the solution is not unique because the input dimension is in general larger than one. Thus one of the infinitely many solutions of (22) is chosen for  $\mathbf{u}_{cor}$  according to (14), (15), or (17). For compensation of small disturbances in legged robot balance control, it is important to choose a corrective motion that is as close as possible to the nominal motion. By weighting, joints can be preferred that do not interfere too much with the nominal task. For example in a walking motion, the acceleration of the arms could be used to correct the ZMP position for small disturbances of the planned motion.

If  $\mathbf{a}^T = \frac{\partial r_{zmp}}{\partial \ddot{\mathbf{q}}}$  becomes zero, the assumption of relative degree one is violated, and thus a solution of (22) is not well-defined any more. In the application, changing the accelerations will not affect the ZMP in such a singular configuration, making control impossible. This problem, nevertheless, will not occur, when only one component of  $\frac{\partial r_{zmp}}{\partial \ddot{\mathbf{q}}}$  vanishes, because then, the other degrees of freedom can still compensate for a lack of controllability in one degree of freedom.

### D. Switching Strategy

The nominal controller acts until the ZMP reaches the boundary of the constraint admissible region. In hardware-implementation also a measurement of ZMP could be used to detect imminent loss of balance. The corrective control law is used when the ZMP reaches the boundary of the admissible region and keeps the ZMP on the boundary. To decide whether the corrective controller may stop acting and the nominal controller can be used again, the first derivative of the ZMP with nominal control is monitored. Only if the derivative points inward the admissible region, the nominal control can be used again. To obtain the ZMP derivative, a virtual model has to be evaluated in parallel to the integration of the real model. See (12) for general formal expressions of this switching control law and Fig. 3 for geometrical illustration.

## V. EXAMPLES

### A. Single Joint Robot

In the first example, a robot is considered that has only one joint and thus only one actuator, see. Fig. 4. Here, the calculation of the appropriate control signal in the ZMP correction mode is unique, compare (13) for  $u \in \mathbb{R}^1$ . As a further simplification, we will consider high values for the parameter  $K$  in the motor dynamics (2) which assumes that the motor dynamics is fast. Thus, as an approximation, the motor dynamics can be neglected, resulting in a simplified dynamical model

$$m_2 l^2 \ddot{\alpha} - m_2 g l \sin \alpha = v, \quad (23)$$

where  $m_2$  is the link mass,  $l$  is the link length,  $\alpha$  is the deflection of the link and  $v$  is the control input. This approximation simplifies the examination of the phase curves, which are 2D in this case instead of 3D.

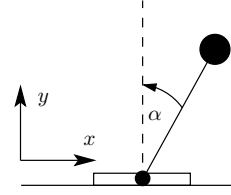


Fig. 4. Single joint robot.

The equation for the ZMP, according to (3) is:

$$r_{zmp} = \frac{m_2 g l \sin \alpha - m_2 l^2 \ddot{\alpha}}{(m_1 + m_2)g - m_2 l (\cos \alpha \dot{\alpha}^2 + \sin \alpha \ddot{\alpha})} \quad (24)$$

Here,  $m_1$  is the mass of the foot. If the foot length is denoted by  $b$ , the constraint for the ZMP is:

$$-\frac{b}{2} < r_{zmp} < \frac{b}{2}$$

For the analysis in simulation experiments, the parameters are chosen:

$b$	0.2 m
$l$	0.4 m
$m_1$	0.3 kg
$m_2$	0.6 kg

In the following the zero dynamics of ZMP control is analyzed for the single joint robot. The zero dynamics is obtained, when the control input  $v$  in (23) is chosen such that the system output  $r_{zmp}$  defined in (24) is constant. Thus, solving (24) for  $\ddot{\alpha}$  yields the zero dynamics

$$\ddot{\alpha} = \frac{((m_1 + m_2)g - m_2 l \cos \alpha \dot{\alpha}^2) r_{zmp} - m_2 g l \sin \alpha}{m_2 l^2 - m_2 l \sin \alpha r_{zmp}}, \quad (25)$$

where  $r_{zmp}$  is a fixed constant position for the ZMP. Equilibrium points  $\alpha^*$  of the zero dynamics (25) solve

$$r_{zmp} = \frac{m_2 l \sin \alpha^*}{m_1 + m_2}.$$

That allows the conclusion that the zero dynamics is in equilibrium state if the ZMP coincides with the center of mass of the robot. The only equilibrium point of interest is the one where the pendulum is above ground and thus  $\alpha$  is in  $[-\frac{\pi}{2}; \frac{\pi}{2}]$ . Equilibrium points do certainly only exist if

$$\left| \frac{r_{zmp}(m_1 + m_2)}{m_2 l} \right| < 1, \quad (26)$$

because  $|\sin \alpha| < 1$ . This inequality has a physical interpretation: The range for the  $x$ -component of the center of mass

$$\frac{m_2 l \sin \alpha}{m_1 + m_2}$$

is bounded by  $-\frac{m_2 l}{m_1 + m_2}$  and  $\frac{m_2 l}{m_1 + m_2}$  and there cannot be an equilibrium solution if the desired ZMP is outside the range where the center of mass can be. See Fig. 5 for geometrical illustration of the equilibrium point.

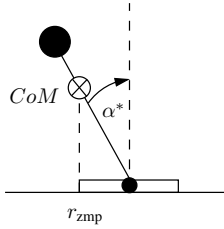


Fig. 5. Equilibrium configuration for single joint robot in ZMP control with  $r_{zmp} = -0.1$  (CoM: Center of Mass).

It is easy to check that the equilibrium point of interest is a saddle point. Figure 6 displays the phase curves for control of the ZMP to  $r_{zmp} = -\frac{b}{2} = -0.1$ . The saddle equilibrium point is taken for a small negative deflection of the link. Further it can be seen that there are no bounded trajectories that do not include penetration of the pendulum with the ground.

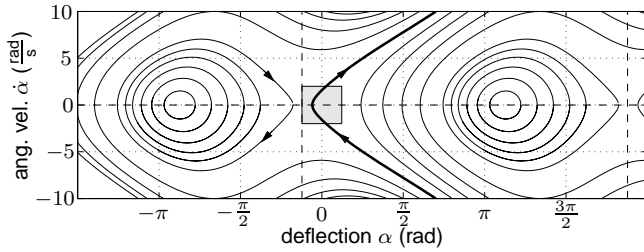


Fig. 6. Phase portrait of the zero dynamics when the ZMP is constantly controlled to the left foot edge. The shaded area marks the section that is displayed in the next figure (Fig. 7).

The switching between control modes, that is characteristic for invariance control, allows to obtain bounded trajectories although the individual control modes result in unbounded behavior. This is illustrated in Fig. 7. The figure displays on the one hand phase curves of the control mode, where the ZMP is kept on the left edge of the foot (solid) and on the other hand phase curves where the ZMP is kept on the right edge of the foot (dashed). By choosing  $\alpha = 0$  as

switching surface, the trajectories remain bounded. Certainly, this holds only in a small region of the phase space.

For this robot configuration, a singularity in the solution of (22) is not possible. This is seen, if  $a = \frac{\partial r_{zmp}}{\partial \alpha} = 0$  is evaluated, or equivalently by investigation of the denominator of (25). The denominator can only be zero if the desired ZMP exceeds the link length of the pendulum. This condition is contained in the condition (26) for the existence of an equilibrium point.

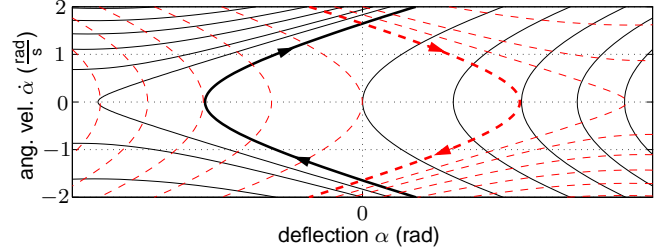


Fig. 7. Phase portrait where the phase curves for control to the left foot edge (solid) intersect with the phase curves for control to the right foot edge (dashed).

## B. Multiple Joint Robot

For the second illustration of invariance ZMP control, the robot from Fig. 1 is considered which presents a more complex scenario for ZMP control due to the larger number of joints. The total weight is assumed to be 76 kg and the height is chosen 1.90 m. The desired motion is balancing on one leg. Therefore an initial posture and a final posture are chosen and a nominal controller drives the robot from initial to final posture. Furthermore at times  $t = 2.0$  s and  $t = 4.0$  s the robot is pushed. The first push is applied frontal at the shoulder, acting instantaneously with 42 Ns, the second push acts from behind hitting the swing foot with 4.2 Ns. The externally applied forces  $\mathbf{F}_{push}$  result in discontinuities in the velocities, calculated from conservation of momentum:

$$\mathbf{M}(\mathbf{q}) (\dot{\mathbf{q}}^+ - \dot{\mathbf{q}}^-) = \mathbf{J}(\mathbf{q})^T \mathbf{F}_{push}$$

The matrix  $\mathbf{J}(\mathbf{q})$  is the Jacobian of the working point of the pushing force. The velocities  $\dot{\mathbf{q}}^-$  and  $\dot{\mathbf{q}}^+$  are velocities immediately before, respectively immediately after the impact. The applied forces are close to the maximum possible forces that can be compensated by the proposed balance control method. It is seen in Fig. 8 that for the execution of the desired motion without corrective controller the ZMP leaves the admissible area, which is the foot contact region between  $y_l = 0$  m and  $y_u = 0.3$  m.

Invariance control with the corrective control law (17) is applied to avoid the loss of balance by consideration of the ZMP constraints in (20) and (21). In the presented simulation, participation of the ankle joint is weighted by  $w_{1,1} = 0.5$  and participation of the shoulder and elbow joint is weighted by  $w_{7,7} = w_{8,8} = 15.0$ . The remaining joints have equal weight one. The corrective control is used, whenever it is detected that the ZMP is about to leave the admissible region. It is switched back to the nominal

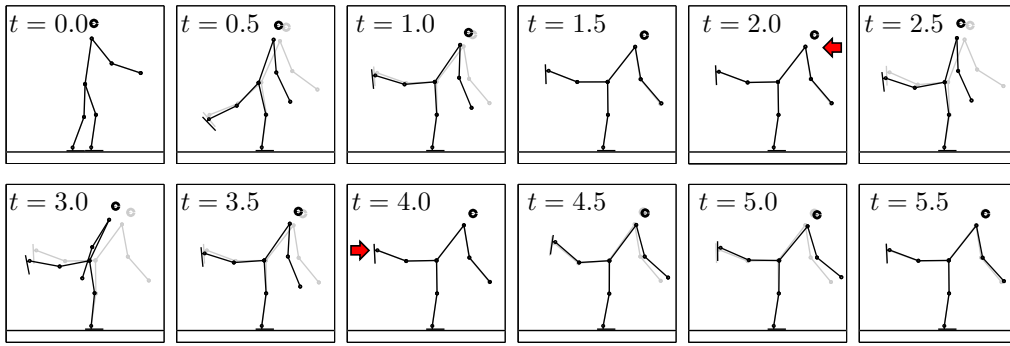


Fig. 9. Snapshot series for balancing task. Comparison of motion without invariance control (grey) and with invariance control (black). Arrows indicate the direction and attack point of the pushing forces at  $t = 2.0$  s and  $t = 4.0$  s.

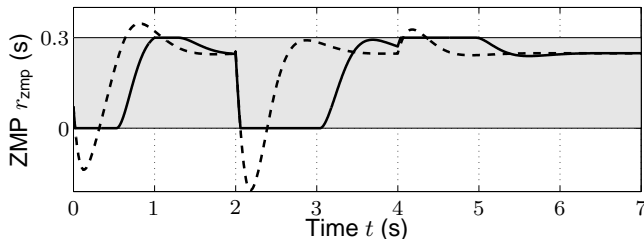


Fig. 8. ZMP trajectory for balancing task. Comparison of ZMP trajectory without invariance control (solid) and with invariance control (dashed). In simulation without invariance control, the ZMP is allowed to leave the admissible region, what, in a real world scenario, would lead to rotation around the foot edges.

controller if the nominal control no longer avoids return to the admissible region. In Fig. 8 the ZMP trajectory with invariance control is compared to the ZMP trajectory without invariance control. In Fig. 9 an associated snapshot series is depicted. In the figures it can be seen that the pushes themselves do not result in large ZMP deflections. It is more the correction motion of the nominal controller that results in large ZMP amplitudes as the correction motion involves sudden high accelerations. The invariance control that keeps the ZMP on the boundary of the admissible region is observed to realize a model-based compliancy in the robot joints.

## VI. DISCUSSION

The example of the single joint robot in Sec. V-A demonstrated how invariance control of the ZMP is also possible without the model refinement by introduction of the motor model. In this setting the ZMP has relative degree  $r = 0$  because it is directly set by the input torques. An extension of the invariance control concept would be necessary to account for outputs with  $r = 0$ . The resulting controller has to solve the algebraic loop and preview the ZMP with nominal controller applied. Nevertheless, for control laws that feed back the ZMP, causality is violated if the model is not refined by the motor dynamics.

The control of the ZMP in general yields a control system with non-minimum phase character. This was shown for the example of the single joint robot yet in Sec. V-A. A

nonlinear control system is defined to be non-minimum phase if it has an unstable zero dynamics. For ZMP control, to maintain a constant position for the ZMP it is necessary to accelerate the masses of the robot, unless the center of mass is equal to ZMP. The necessary accelerations increase the distance between ZMP and center of mass. Therefore persistent control of the ZMP yields an unstable system and only a switching scheme between nominal controller and invariance controller, as proposed, makes it possible to have an overall stable control system. A proof of stability is subject of future research. The stability analysis will also provide insight into limitations for balance control. Like humans, a robot cannot compensate arbitrary disturbances just by adequate acceleration of its masses. If disturbances are too high, backup strategies have to be available, e.g. making contact with the other foot to enlarge the support area. An important issue in balance control is to find criteria that indicate that ZMP control will not succeed any more and to determine the time when to switch to a backup strategy.

## VII. CONCLUSION

The paper presented a balance control approach for legged robots realized by invariance control. Balance control avoids free rotation around foot edges. The ZMP is confined to an admissible region by switching between two control modes: the nominal control inside the admissible region and the corrective control on the boundary of the admissible region. In the calculation of the corrective control signal, the distance to the nominal control signal is minimized. An additional weighting term in the minimization allows to select which joints participate in the correcting motion. The presented approach is seen as general and is not restricted to special robot structures.

## VIII. ACKNOWLEDGMENTS

Thanks to C. Raubitschek and T. Gern for preliminary studies.

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