Initialization Concepts for Optimal Control of Hybrid Systems

Benjamin Passenberg * **Magnus Kröninger** *
Georg Schnattinger ** **Marion Leibold** * **Olaf Stursberg** ***
Martin Buss *

* *Institute of Automatic Control Engineering, Technische Universität München, Munich, Germany, (e-mail: {passenberg, marion.sobotka,mb}@tum.de, magnus.kroeninger@mytum.de).*
** *Institute for High-Frequency Engineering, Technische Universität München, Munich, Germany, (e-mail: georg.schnattinger@tum.de).*
*** *Institute of Control and System Theory, University of Kassel, Kassel, Germany, (e-mail: stursberg@uni-kassel.de).*

Abstract: Indirect methods solve optimal control problems for hybrid systems with high precision, but they are difficult to initialize. To overcome the initialization difficulties, two different concepts based on direct methods are presented. In the first approach, the hybrid optimal control problem is solved by a direct method until the precision is sufficient for a successful initialization of the indirect method. The second approach decomposes the hybrid optimal control problem into non-hybrid subproblems, where each subproblem can be initialized separately by a direct method. This results in a significantly higher robustness of the initialization compared to the first approach. However, the precision of the solution with the indirect method achieved in the first approach is higher. The two concepts are compared in a numerical example.

Keywords: optimal control, hybrid systems, initialization, indirect optimization methods, autonomous vehicles

1. INTRODUCTION

Direct and indirect methods are prominent numerical methods for finding an optimal control for nonlinear systems [Bock and Plitt, 1984, von Stryk and Bulirsch, 1992, Betts, 1998, Sargent, 2000]. Indirect methods provide highly accurate solutions, but they are difficult to initialize. Two-stage approaches are known from the literature, where a direct method solves the optimal control problem first and the solution is used to initialize an indirect method, which improves the solution. This initialization idea is extended to hybrid systems in this paper. Here, hybrid systems with autonomous switching are considered, where the sequence of discrete states is fixed and the continuous dynamics switches whenever the continuous state trajectory hits a switching manifold.

Direct methods transform the infinite dimensional problem of finding optimal state and control trajectories into a finite dimensional, nonlinear programming problem by evaluating state and control values only at a finite number of time samples. By varying the state and control values at those samples, the cost criterion is directly optimized which can be performed by nonlinear programming methods like sequential quadratic programming. The nonlinear program is usually formulated as a direct multiple shooting algorithm [Bock and Plitt, 1984, Betts, 2001] or as a direct collocation algorithm [von Stryk, 1993]. Direct methods show several advantages compared to indirect ones. Direct methods are easier to initialize due to a larger domain of convergence and the physically intuitive meaning of the optimization variables. Furthermore, the sequence of constrained and unconstrained arcs does not need to be guessed in advance but evolves during the optimization. Constrained and unconstrained arcs are segments of the state trajectory without active or with the same active state constraints, respectively. However, direct methods deliver less precise results, mainly due to discretization errors. Additionally, direct methods sometimes converge to local minima, which are introduced by the discretization and which do not match a true locally optimal solution [von Stryk and Bulirsch, 1992].

In contrast, indirect methods solve a multi-point boundary-value problem (MPBVP) that originates from necessary optimality conditions which an optimal solution has to satisfy. The optimality conditions are derived by calculus of variations [Bryson and Ho, 1975] or Pontryagin's minimum principle [Pontryagin et al., 1963]. Common methods to solve the MPBVP are given by gradient methods [Gottlieb, 1967, Bryson and Ho, 1975] and indirect multiple shooting [Bulirsch, 1971, Bock, 1978, Riedinger et al., 2005]. Advantages of indirect methods are the high precision of a solution, the fast convergence near the optimal solution and the reliability. Disadvantages are the relatively small domain of convergence and the physically non-intuitive nature of adjoint variables arising from the optimality conditions. This causes the initialization process to be difficult. Additionally, the sequence of state constrained and unconstrained arcs has, in general, to be known beforehand.

The idea of combining the advantages of the two approaches has already been realized in [von Stryk and Bulirsch, 1992, Grimm and Markl, 1997, Seywald and Kumar, 1996]. Basically, this is developed for non-hybrid systems. In principle, a direct method is applied to initialize an indirect one and information about the sequence of state constrained and unconstrained arcs is gained from the direct approach. In [von Stryk and Bulirsch, 1992] and [Grimm and Markl, 1997], a direct collocation and a direct multiple shooting method are used to determine initial values and the sequence of constraints for an indirect multiple shooting method. Estimates of the adjoint variables are obtained directly from the applied numerical optimization algorithms. Furthermore, it is proved in [Grimm and Markl, 1997] that those estimates converge to the exact optimal adjoint values if the discretization is continuously refined. In [Seywald and Kumar, 1996], the fact is exploited that adjoint variables form the sensitivity of the performance with respect to the initial state. Thus, solving a series of optimization problems with perturbed initial state by direct methods provides the desired initial values for the adjoint variables.

The aim of this paper is to introduce and discuss two initialization concepts for the indirect solution of hybrid optimal control problems (HOCPs) with autonomous switching. In this paper, the sequence of discrete states is considered to be fixed and is not subject to optimal control. An algorithm, which optimizes the discrete state sequence on an upper layer, has been introduced in Passenberg et al. [2010b] and can be initialized with the second concept. In the first concept, a direct and an indirect multiple shooting method solve the HOCP entirely. First, the direct method is applied and delivers estimates of the state and adjoint variables. As those estimates of the adjoint variables do not match the corresponding values in the indirect method on state constrained arcs, a backward integration scheme is applied to determine a suitable initialization. The first concept suffers from the complexity of the HOCP and the required precision of adjoint variables to successfully initialize the indirect method, what can lead to cases of non-successful initialization. Therefore, a second initialization concept is introduced, where the HOCP is decomposed into non-hybrid subproblems. Each of those subproblems is solved separately from the others, at first, by a direct method for non-hybrid systems. Secondly, each subproblem is solved by an indirect method for non-hybrid systems based on an initialization with the solution of the direct method. This simplifies the initialization significantly due to the reduced complexity of solving subproblems instead of the entire HOCP at once. The subproblems are connected by switching points and times, which are varied until the entire HOCP satisfies all necessary optimality conditions. An algorithm using the decomposition technique is presented in [Passenberg et al., 2010b].

The paper is structured as follows: In Sec. 2, the HOCP is formulated and Sec. 3 introduces and discusses two initialization concepts for the solution of the HOCP. In Sec. 4, both concepts are applied to a numerical example and Sec. 5 concludes the paper.

2. PROBLEM FORMULATION

2.1 HOCP

Consider the HOCP with autonomous switching:

$$\min_{u\in\mathcal{U}} J = \min_{u\in\mathcal{U}} \Big(g(x(t_f)) + \sum_{i=0}^{K} \int_{t_i}^{t_{i+1}} \phi_i(x(t), u(t))\, dt \Big) \quad (1)$$

with

$$\dot{x} = f_i(x(t), u(t)) \quad \text{a.e. } t \in [t_i, t_i + 1) \quad (2)$$
$$0 = m_{i-1,i}(x(t_i)) \quad \text{for } i \neq 0 \quad (3)$$
$$0 = \psi(x(t_f)) \quad (4)$$
$$0 \geq h_i(x(t)) \quad (5)$$
$$x(t_0) = x_0\,,\ i(t_0) = 0\,, \quad (6)$$

where $t \in \mathbb{R}^+$, $x(t) \in \mathcal{X}_i \subseteq \mathbb{R}^{n_x}$ for $t \in [t_i, t_{i+1})$, $u(t) \in U_i \subseteq \mathbb{R}^{n_u}$ for $t \in [t_i, t_{i+1})$, and $i \in \{0, 1, ..., K\}$ are time, continuous state, control variable, and discrete state. The functions $g : \mathbb{R}^{n_x} \to \mathbb{R}$, $\phi_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$, $f_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$, $m_{i-1,i} : \mathbb{R}^{n_x} \to \mathbb{R}$, $\psi : \mathbb{R}^{n_x} \to \mathbb{R}^{n_f}$, and $h_i : \mathbb{R}^{n_x} \to \mathbb{R}^{n_h(i)}$ are terminal costs, running costs, system dynamics, switching manifolds, terminal conditions, and inequality constraints for the continuous state, where $n_x = \dim(x)$, $n_u = \dim(u)$, $n_f = \dim(\psi)$, $n_h(i) = \dim(h_i)$, and $K \in \mathbb{N}$. A subscript i denotes the dependence of a function on discrete state i. It is assumed that all functions are sufficiently smooth with respect to their arguments for the analysis in this paper. $\mathcal{U}$ is the collection of all sets $\mathcal{U}_i$ denoting all measurable and bounded functions u taking values in U_i. It is assumed that the boundary of each set $\mathcal{X}_i$ can be expressed by a function h_i with finite dimension $n_h(i)$. Autonomous switches from discrete state i to $i+1$ are triggered, when the continuous state trajectory x hits the switching manifold $m_{i,i+1}(x(t_{i+1})) = 0$. It is assumed that switching manifolds are unbounded and do not intersect. Here, $t_{K+1} = t_f$ and the last interval $[t_K, t_{K+1}]$ is closed. It is assumed that system executions are unique and that optimal state and control trajectories exist.

2.2 Optimality Conditions

The Hamiltonian function

$$H_i(x(t), u(t), \lambda(t)) = \phi_i(x(t), u(t)) + \lambda^T(t) f_i(x(t), u(t)) + \mu^T(t) h_i^{(p),a}(x(t), u(t)) \quad (7)$$

is defined with the time-varying multipliers $\lambda(t) \in \mathbb{R}^{n_x}$ and $\mu(t) \in \mathbb{R}^{n_a(i)}$, where $n_a(i)$ is the number of active inequality constraints, i.e. of constraints for which applies $h_i^a(x(t)) = 0$. Let for a simplified notation $n_a(i)$ be equal to 1. The extension to multi-dimensional state constraints is straight forward. The control $u(t)$ appears in the p-th time derivative of an active inequality constraint h_i^a for the first time with $p \in \mathbb{N}_0$ and this derivative $h_i^{(p),a}$ is added to (7). All other derivatives $h_i^{(0),a}, ..., h_i^{(p-1),a}$ are combined in the vector

$$S_i(x(t)) = \begin{pmatrix} h_i^a(x(t)) \\ h_i^{(1),a}(x(t)) \\ \dots \\ h_i^{(p-1),a}(x(t)) \end{pmatrix}. \quad (8)$$

Note that for convenience, the dependence of all variables on time t are not explicitly stated anymore. Trajectories

x^* and u^* that minimize the cost (1) under consideration of (2) - (6) fulfill the following conditions [Bryson and Ho, 1975, Pontryagin et al., 1963, Sussmann, 1999, Shaikh and Caines, 2007]:

$$\dot{\lambda} = -\nabla_x H_i \quad \text{a.e. } t \in [t_i, t_i + 1) \tag{9}$$
$$\lambda(t_f) = \nabla_x g(x(t_f)) + \nabla_x \psi^T(x(t_f))\,\nu \tag{10}$$
$$\lambda(t_i^-) = \lambda(t_i) + \nabla_x m_{i-1,i}^T(x(t_i))\,\pi_i \quad \text{for } i \neq 0 \tag{11}$$
$$H_{i-1}(t_i^-) = H_i(t_i) \quad \text{for } i \neq 0 \tag{12}$$
$$\lambda(t_l^-) = \lambda(t_l) + \nabla_x S_i^T(x(t_l))\,\kappa_l \tag{13}$$
$$H_i(t_l^-) = H_i(t_l) \tag{14}$$
$$\mu^T h_i = 0 \tag{15}$$
$$\mu \geq 0 \tag{16}$$
$$u^* = \arg\min_{u \in U} H_i(x, u, \lambda) \tag{17}$$

with $i \in \{0, 1, ..., K\}$, $l \in \{1, ..., L\}$, K the number of autonomous switchings, L the number of inequality constraints met, $\nu \in \mathbb{R}^{n_f}$, $\pi_i \in \mathbb{R}^{n_m(i-1,i)}$, $\kappa_l \in \mathbb{R}^p$, $\lim_{t \to t_i, t<t_i} t =: t_i^-$ and $\lim_{t \to t_l, t<t_l} t =: t_l^-$. Conditions (15) - (17) must hold for all $t \in [t_i, t_{i+1})$. Note that there might exist cases where (12) cannot be satisfied [Sussmann, 1999, Passenberg et al., 2010a].

3. INITIALIZATION

Initializing an indirect multiple shooting algorithm for solving a HOCP is comparably difficult to the case of non-hybrid optimal control problems with state constraints, which allows the adjoint variables to jump. The difficulty arises from the physically non-intuitive adjoint variables and the small domain of convergence. The latter is mainly due to integrating the adjoint differential equations in the instable direction, which turns the optimization problem ill-conditioned. Additionally, good estimates of the switching times and states are required in the fixed sequence of discrete states.

In the following, a direct and an indirect multiple shooting algorithm for solving HOCPs are explained shortly as they form the basis for two initialization concepts introduced later in the section.

3.1 Direct Multiple Shooting

The continuous time HOCP is time discretized with time samples t_s, $s \in \{0, 1, ..., N_s\}$, $t_{N_s} = t_f$, to transform the HOCP into a nonlinear program [Bock and Plitt, 1984]. For the nonlinear program, the continuous state $x[s] := x(t_s)$ and control variables $u[s] := u(t_s)$ for the samples s form the optimization variables. From the set of samples $s \in \{0, 1, ..., N_s\}$, so called key samples s_i are picked and are associated with the initial, hybrid switching, and final time instants t_i, $i \in \{0, 1, ..., K, K+1\}$, such that $x[s_i] := x(t_i)$. As the time instants t_i of hybrid switches will vary during the optimization, it is necessary to define further optimization variables $\Delta t_i = t_{i+1} - t_i$, $i \in \{0, 1, ..., K\}$, for the duration of being in discrete state i. Equally distributed inter-sampling times $\Delta t[s] := t_{s+1} - t_s = \frac{\Delta t_i}{s_{i+1} - s_i}$ are determined for all samples $s \in \{s_i, ..., s_{i+1} - 1\}$ relative to Δt_i. The solver for the nonlinear program varies $x[s]$, $u[s]$, and Δt_i such that the discretized cost functional

$$J = g[N_s] + \sum_{i=0}^{K} \Omega_i \tag{18}$$

with

$$\Omega_i = \frac{1}{2}\Delta t[s_i]\phi_i[s_i] + \frac{1}{2}\Delta t[s_{i+1}]\phi_i[s_{i+1}] + \sum_{s=s_i+1}^{s_{i+1}-1} \frac{1}{2}(\Delta t[s-1] + \Delta t[s])\phi_i[s] \tag{19}$$

is minimized with discretized equality and inequality constraints:

$$0 = \frac{1}{2}(f_i[s+1] + f_i[s]) - \frac{x[s+1] - x[s]}{\Delta t[s]} = \gamma[s] \quad \text{for } s \in \{s_i, s_{i+1} - 1\} \tag{20}$$
$$0 = m_{i-1,i}[s_i] \quad \text{for } i \neq 0 \tag{21}$$
$$0 = \psi[N_s] \tag{22}$$
$$0 = x[0] - x_0 \tag{23}$$
$$0 = t_f - t_0 - \sum_{i=0}^{K} \Delta t_i \tag{24}$$
$$0 \geq h[s] \tag{25}$$
$$0 \geq c[s] \tag{26}$$
$$0 \geq -\Delta t_i\,. \tag{27}$$

Here, $c[s]$ is a set of functions approximating the boundaries of the set U for the control $u[s]$. A function followed by $[s]$ means that the function is evaluated with all its arguments at time t_s. Instead of a trapezoidal approximation of the system dynamics (20) any other method like Runge-Kutta or a direct integration of the differential equations (2) can be used if desired. Note that in this formulation, the inequality constraints (25) and (26) are only required to be satisfied at the sampling times t_s. There is no need to prespecify the sequence or times of active inequality constraints. It is found during the optimization. Depending on the complexity of the HOCP, the necessary accuracy of the initialization for a successful solution with the direct method varies but is in general far from the accuracy required by indirect methods. For HOCPs with low complexity, an initialization with zero state $x[s]$ and control $u[s]$ values and equally distributed time intervals Δt_i is often enough.

For solving the nonlinear program, a suitable solver adjoins the equality and inequality constraints (20) - (27) with multipliers to the costs (18). The multipliers for adjoining the system dynamics (20), the state constraints (25), and the interior and terminal constraints are called $\lambda[s]$, $\mu[s]$, $\pi[s_i]$, and $\nu[N_s]$, respectively. It can be shown that for $N_s \to \infty$ the costs of the time discretized optimal control problem (18) converge to the costs of the originial HOCP (1) [Goh and Teo, 1988]. Furthermore, the sampled multipliers converge for $N_s \to \infty$ to the optimal multipliers $\lambda^*(t_s)$, $\mu^*(t_s)$, $\pi^*(t_i)$, and $\nu^*(t_f)$ [Grimm and Markl, 1997] except of $\lambda[s]$ for active inequality constraints. Thus, the sampled multipliers provided by the solver can in general be used as estimates for the optimal multipliers.

3.2 Indirect Multiple Shooting

The goal of indirect methods is to solve the MPBVP resulting from the optimality conditions given in Sec. 2

[Bock, 1978, Riedinger et al., 2005]. Before solving the MPBVP, the sequence of hybrid switches, constrained, and unconstrained arcs has to be pre-specified. A set of time instants t_r, $r \in \{1, 2, ..., N_r\}$ is chosen with $t_0 < t_1 < ... < t_{N_r} < t_f$ where N_r is large enough to assign one t_r to each autonomous switching instant t_i and constraint entry time t_l. At each time t_r, values for the continuous state x_r and the adjoint variable λ_r have to be guessed. Those values are used as initialization for integrating the system and adjoint differential equations (2) and (9) with the control $u(t)$ given by (17) from t_r to t_{r+1}^- to obtain the state and adjoint variables $x(t_{r+1}^-)$ and $\lambda(t_{r+1}^-)$. This enables us to define a set of error equations, which a solver steers simultaneously to zero by varying the initial values x_r and λ_r. The error equations are set up depending on if a hybrid switch (28), an entry onto a state constraint arc (29), or neither the first nor the second case (30) occurs at time t_r:

$$e_r = \begin{pmatrix} x(t_r^-) - x_r \\ \lambda(t_r^-) - \lambda_r - \nabla_x m_{i-1,i}^T(x(t_r^-))\pi_i \\ H_{i-1}(t_r^-) - H_i(t_r) \\ m_{i-1,i}^T(x(t_r^-)) \end{pmatrix} \tag{28}$$

$$e_r = \begin{pmatrix} x(t_r^-) - x_r \\ \lambda(t_r^-) - \lambda_r - \nabla_x S_i^T(x(t_r^-))\kappa_l \\ H_i(t_r^-) - H_i(t_r) \\ S_i^T(x(t_r^-)) \end{pmatrix} \tag{29}$$

$$e_r = \begin{pmatrix} x(t_r^-) - x_r \\ \lambda(t_r^-) - \lambda_r \end{pmatrix}, \tag{30}$$

respectively, for $r \in \{1, ..., N_r\}$. Additionally, the terminal error with dimension n_x is given by:

$$e_f = \begin{pmatrix} \psi(x(t_f)) \\ \lambda(t_f) - \nabla_x g(x(t_f)) - \nabla_x \psi^T(x(t_f))\,\nu \end{pmatrix}. \tag{31}$$

There exist various possibilities how to determine π_i, κ_l, ν, and t_r, e.g. the parameters can also be treated as optimization variables. At the initial time t_0, only an adjoint optimization variable λ_0 is introduced since the state variable is set to the known value x_0. The error equations are usually solved by a modified Newton-method for root finding.

3.3 Concept 1

In the first concept, a complete solution of the HOCP is obtained by the direct multiple shooting algorithm in Sec. 3.1 and is used to initialize the indirect multiple shooting algorithm presented in Sec. 3.2. The first step of the initialization of the indirect method is to specify the sequence of autonomous switches, constrained and unconstrained arcs. The sequence of constraints is taken from the values of the multipliers $\mu[s]$ provided by the solution of the direct method and is fitted into the prespecified hybrid sequence.

Next, the optimization horizon of the indirect method has to be divided into multiple intervals. The most intuitive way is to set $t_r = t_s$. However, the number of intervals N_r in the indirect method may also be chosen significantly lower than the number of samples N_s in the direct method. The domain of convergence of the indirect method is relatively small, thus it is essential to have good estimates for the initial values x_r and λ_r of each interval (e.g. adjoint estimates must equal the optimal values for at least two digits for the system discussed in Bulirsch et al. [1991]). In the following, a concept for obtaining estimates x_r and λ_r from the solution of the direct multiple shooting is shown. If no state inequality constraints appear, the values x_r and λ_r can be calculated with $x[s]$ and $\lambda[s]$. However, for active state inequality constraints, the optimal adjoint variables are not unique in the subspace $S_i(x(t))$. This implies that the values $\lambda[s]$ may be completely different from the optimal adjoint variables $\lambda^*(t_s)$ in the subspace $S_i(x(t_s))$. Only the uniquely determined part of $\lambda[s]$, which is in the complementary subspace $\bar{S}_i(x(t_s))$, is used for updates. The other part is obtained by a backward integration of previously and uniquely known estimates of the adjoint variables. To distinguish between the uniquely determined part of λ and the non-unique part, projections into the non-unique and the unique subspace are applied

$$P_{B(S_i)} = B(S_i)\, B(S_i)^T \tag{32}$$

$$P_{\bar{B}(S_i)} = I - P_{B(S_i)}, \tag{33}$$

respectively, with the identity matrix I. The projections are based on $B(S_i)$, which is a normalization and an orthogonalization of the subspace $S_i(x(t_s))$ that can be achieved for example by a Gram-Schmidt process [Watkins, 2002].

As the adjoint variable $\lambda[s]$ is related to the equality constraint $\gamma[s]$, $\lambda[s]$ corresponds to its continuous-time equivalent $\lambda(\bar{t}_s)$ with $\bar{t}_s = \frac{1}{2}(t_{s+1} + t_s)$. At each time $\bar{t}_s$, the backward integration of Eq. (2) and (9) is updated according to

$$\hat{x}(\bar{t}_s) = \frac{1}{2}(x[s] + x[s+1]) \tag{34}$$

$$\hat{\lambda}(\bar{t}_s) = P_{\bar{B}(S_i)}\left[\alpha\lambda[s] + (1-\alpha)\tilde{\lambda}(\bar{t}_s)\right] + P_{B(S_i)}\tilde{\lambda}(\bar{t}_s), \tag{35}$$

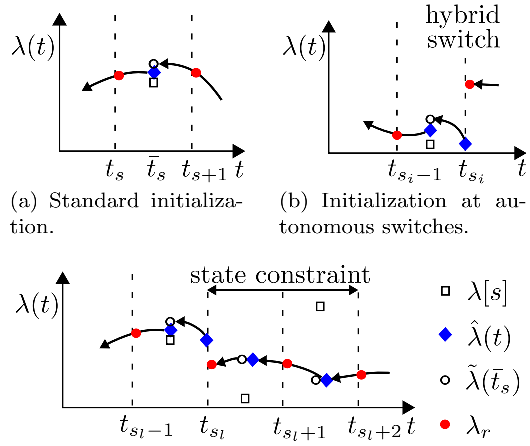
where the uniquely determined part of the adjoint variable is chosen as a convex combination of the values $\lambda[s]$ from the direct solution and $\tilde{\lambda}(\bar{t}_s)$ from the backward integration with $\alpha \in \mathbb{R}, \alpha \in [0, 1]$, see Fig. 1(a). In contrast, the state $x(\bar{t}_s)$ is always updated only by the values $x[s]$ and $x[s+1]$ from the direct method due to their good quality. At the final time t_f, at the switching times $t_i = t_{s_i}$, see Fig. 1(b), and at the entry times of constrained arcs $t_l = t_{s_l}$, see Fig. 1(c), the backward integration is initialized by

$$\hat{\lambda}(t_f) = \nabla_x g(x[N_s]) + \nabla_x \psi^T(x[N_s])\,\nu[N_s] \tag{36}$$

$$\hat{\lambda}(t_i^-) = \tilde{\lambda}(t_i) + \nabla_x m_{i-1,i}^T(x[s_i])\,\pi[s_i] \tag{37}$$

$$\hat{\lambda}(t_l^-) = P_{\bar{B}(S_i)}\tilde{\lambda}(t_l) + P_{B(S_i)}\lambda[s_l - 1] \tag{38}$$

as no adjoint variables $\lambda[N_s]$, $\lambda[s_i^-]$, and $\lambda[s_l^-]$ can be provided by the direct solver. The samples s_i and s_l denote an autonomous switching and an entry into a state constraint. Here, the values $\tilde{\lambda}(t_i)$ and $\tilde{\lambda}(t_l)$ are determined by the update (35) and a backward integration from $\bar{t}_{s_i}$ to t_i and from $\bar{t}_{s_l}$ to t_l, respectively. The multipliers $\nu[N_s]$ and $\pi[s_i]$ are given by the direct method. As no multiplier $\kappa[s_l]$ can be obtained from the solver, $\lambda[s_l^-]$ can only be approximated by $P_{B(S_i)}\lambda[s_l - 1]$, which uses the neighboring sample $s_l - 1$. In the scalar example shown in Fig. 1(c), the new starting values $\hat{\lambda}(\bar{t}_{s_l})$ and $\hat{\lambda}(\bar{t}_{s_l+1})$ can only be determined by the backward integration according to (35) while a state constraint is active. Note that for unique updates the update rule (35) requires the assumption that two overlapping state constraints cause jumps in the adjoint variables in orthogonal subspaces S_i.



(a) Standard initialization.

(b) Initialization at autonomous switches.

(c) Initialization for state constraints.

Fig. 1. Examples of the initialization process with backward integration to find initial adjoint variables λ_r for the indirect method from the solution of the direct method $\lambda[s]$. Vertically aligned groups of points have different values, while horizontally aligned groups of points denote equal values.

3.4 Concept 2

The second concept is based on an algorithm presented in [Passenberg et al., 2010b]. There, the HOCP (1) is decomposed into a two-layered problem. On the higher level, a sequence of switching points $x(t_i)$ and entry $x(t_l)$ and exit points $x(t_l^{\text{out}})$ of state constraint arcs is moved along the corresponding switching manifolds and state constraints with a gradient method until the optimality conditions (11) - (14) are satisfied. On the lower level, optimal control subproblems with constant discrete state and constant active state constraints are solved from one point to the next in the sequence of switching points and entry and exit points of state constraints. The optimal control of the subproblems is performed by indirect multiple shooting as introduced in Sec. 3.2. Here, only the error Eq. (30) and (31) are considered as the subproblems are non-hybrid and have constant active state constraints.

For the initialization of the algorithm, a sequence of discrete states and corresponding switching points is guessed. This leads to optimal control subproblems with constant discrete state but possibly varying active state constraints between two consecutive switching points $x(t_i)$ and $x(t_{i+1})$. Then, each subproblem is solved separately from the others by the direct method from Sec. 3.1, where the costs in (18) reduces to $J = \Omega_i$ for subproblem i defined between the samples s_i and s_{i+1}. Furthermore, (21) is replaced by the boundary conditions $x(t_i) - x[s_i] = 0$ and $x(t_{i+1}) - x[s_{i+1}] = 0$. Thus, the entire HOCP does not need to be solved at once by the direct optimization method. From the solution of the direct method, estimates of the optimal state and adjoint variables and the sequence of constrained and unconstrained arcs with estimates of the optimal entry and exit points are obtained for each subproblem. The estimates of the sequence of constrained and unconstrained arcs are used to divide the subproblems further into subproblems with constant discrete states and constant active state constraints. These new subproblems are now solved individually by indirect multiple shooting with the initial estimates of the state and adjoint variables from the direct method. These estimates are determined with the backward integration scheme introduced in Concept 1. Varying the switching, entry and exit points to find the optimal state and control trajectory of the entire HOCP problem is performed by the algorithm introduced in [Passenberg et al., 2010b]. Note that the algorithm there is set up for hybrid systems with partitioned state spaces, so that the optimal sequence of discrete states can be found during the optimization and does not need to be known a priori. However, by removing the algorithm's ability to change the discrete state sequence during the optimization, the algorithm can be applied to the more general class of hybrid systems specified in Sec. 2.

3.5 Comparison

In the following, a short discussion of the advantages and disadvantages of the two concepts is given. Concept 1 delivers solutions with higher precision than concept 2. This means that for the same number of iterations and comparable initial values, the solution of concept 1 is closer to the optimal solution. Solving the complete HOCP at once, the direct method in concept 1 allows to apply indirect multiple shooting for the solution of the entire HOCP, which achieves a very high precision with its Newton-iterations. In contrast, the upper layer of the indirect approach in concept 2, which shifts the switching points, uses updates based on first-order gradients. In general, algorithms based on first-order gradient approaches show a lower precision for a comparable number of iterations and a larger domain of convergence than algorithms with Newton methods. The latter argument directly leads to the major disadvantage of concept 1, which is the lower robustness. This means the frequency of failing in the initialization is higher. An initialization fails if the indirect method does not converge to an optimal solution. In such a case, the precision of the solution from the direct method has to be increased, e.g. by augmenting the number of sampling points or refining the initialization of the direct method. In concept 1, a very complex optimization problem is to be solved, which causes difficulties in achieving the required precision for initializing the indirect multiple shooting with Newton-iterations successfully. The gradient method applied in concept 2 is capable of converging to the optimal solution, even if the initial switching points are far from the optimal switching points. Furthermore, the first approach needs a higher precision in the estimates of the optimal state and adjoint state variables. This is due to the influence of small errors of the initial values compared to the optimal ones on the complete HOCP. In contrast, the impact of initialization errors in concept 2 remains bounded to each subproblem. Additionally, the precision of the estimates is higher and the estimates can be found faster as the decomposition into non-hybrid and smaller subproblems simplifies the solution for the direct method. The direct method in concept 2 tolerates a less accurate initialization due to the lower complexity of the optimal control subproblems in contrast to the complexity of the entire HOCP. In total, this leads to a higher robustness of the initialization in concept 2.

Consider the case of hybrid systems with partitioned state space and unspecified discrete state sequence. There, concept 2 can be initialized with some feasible sequence of discrete states and during the optimization the sequence is varied until a locally optimal sequence is found [Passenberg et al., 2010b]. In comparison, the direct method in concept 1 either remains unchanged, such that it searches for an optimal trajectory in the prespecified hybrid sequence. Then, the optimal sequence has to be found by solving a HOCP for all possible sequences, which exhibits combinatoric computational complexity. Alternatively, the direct method could be modified such that it finds the locally optimal sequence automatically, but here it is especially difficult to reach the necessary precision for the initialization.

4. NUMERICAL EXAMPLE

4.1 Autonomous Vehicle

Both initialization concepts are used for optimizing a drive-around maneuver of an autonomous car. The goal is to find the optimal state and control trajectory to drive around a parked car. The autonomous car is modeled as a *unicycle* or *Dubin's vehicle* [Sanfelice and Frazzoli, 2008]:

$$\dot{x} = v\cos\theta \qquad (39)$$
$$\dot{y} = v\sin\theta \qquad (40)$$
$$\dot{\theta} = u \qquad (41)$$
$$|u| \leq 0.8\,, \qquad (42)$$

where (x, y) is the position of the car, θ the orientation, v the constant velocity, and the steering input u is the velocity of a change in the orientation. The problem becomes hybrid by the model of the environment and the associated cost functions. A two lane road is modeled with three discrete states $q = \{1, 2, 3\}$, such that if the center point of the car is in discrete state $q = 1$ or $q = 3$, then the whole car is completely on the right or left lane. The transition region for the center point of the car between the right and the left lane is discrete state $q = 2$, which corresponds to driving in the middle of the road, see Fig. 2:

$$q = 1: \quad -2.15 \leq y \leq -0.85 \qquad (43)$$
$$q = 2: \quad -0.85 \leq y \leq 0.85 \qquad (44)$$
$$q = 3: \quad 0.85 \leq y \leq 2.15\,. \qquad (45)$$

The hybrid formulation provides the advantage that clear decisions to drive on the right, middle, or left part of the road are made by the autonomous car. Furthermore, by the choice of the assigned cost functions

$$\phi_1 = c_1 u^2 + c_2(y + 1.5)^4 \qquad (46)$$
$$\phi_2 = c_1 u^2 + c_3 \qquad (47)$$
$$\phi_3 = c_1 u^2 + c_4\,, \qquad (48)$$

the car can be manipulated to drive on the right lane except for special situations like another car blocking the right lane. A car blocking the right lane is modeled with two state inequality constraints, which demand the autonomous car to drive around the obstacle on the left lane, such that no collision can occur. Further state inequality constraints are represented by the right border of the right lane and the left border of the left lane.

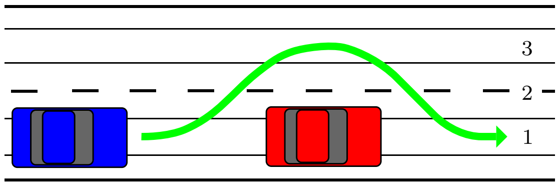


Fig. 2. Autonomous car driving around a parked car in a division of the road into three discrete states.

The optimization task is to find the optimal state and control trajectory around the parked car with discrete state sequence $[1, 2, 3, 2, 1]$. The optimal solution minimizes the costs accumulated by (46) - (48) over a time horizon of 6 sec. under consideration of the car dynamics (39) - (41), the control (42), and state constraints. Here, the constants are chosen to be: $v = 15$, $c_1 = 20$, $c_2 = 10$, $c_3 = 5$, and $c_4 = 10$.

4.2 Results

The results of optimizing with concepts 1 and 2 are illustrated in Fig. 3 and Fig. 4, respectively. Achieving these results, the strengths and weaknesses of both concepts as discussed in Sec. 3.5 are observed. The precision of the solution of concept 1 was higher, which can be seen in the optimality conditions, where each condition is satisfied in concept 1 to a precision of at least 10^{-3} compared to $7{\cdot}10^{-1}$ in concept 2. Consequently, the costs of the reached optima differ with $J = 17.34$ in concept 1 and $J = 17.68$ in concept 2. The costs found by the direct method in concept 1 are $J = 26.39$. Our experience shows that the initialization with concept 2 is more robust and needs less tuning of optimization parameters to achieve convergence. Additionally, the initialization for the direct method in the individual discrete states may be less accurate. The computation time was 4.5 h in concept 1 and 15.1 h in concept 2. The computations were performed on a AMD 3 GHz processor running Windows and Matlab 2009a.

5. CONCLUSION

Two concepts are introduced for initializing indirect multiple shooting methods with direct methods for the optimal control of hybrid systems. In concept 1 the whole optimization problem is initialized directly, whereas in concept 2 the problem is decomposed such that only subproblems have to be initialized. For strongly nonlinear systems with many hybrid switches, concept 2 shows the advantage of being an initialization scheme, which is more robust against inaccuracies of the solution from the direct method. If a high precision of the solution is desired, it is recommended to apply concept 1. It is even possible to use concept 2 for initializing concept 1, such that HOCPs can be solved robustly and with high precision. In future, both concepts will be tested with more complex examples and methods will be explored how an initial discrete state sequence can be found in the neighborhood of the optimal sequence.

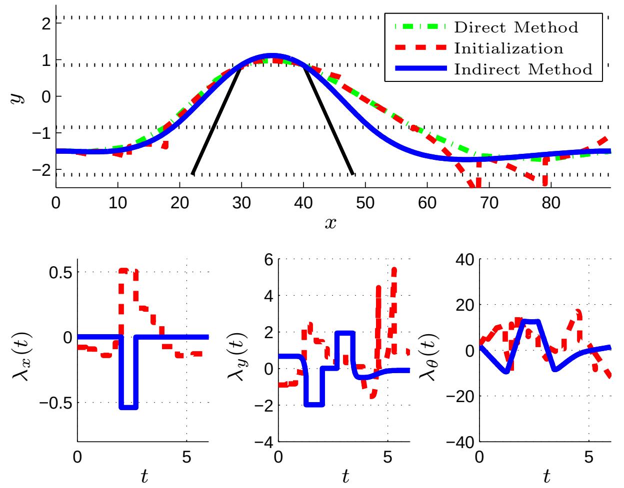


Fig. 3. Comparison of the trajectories found by concept 1 after optimizing with the direct method (green dotted line), after initializing the indirect method with the direct one (red dashed line), and after optimizing with the indirect one after the initialization (blue solid line).

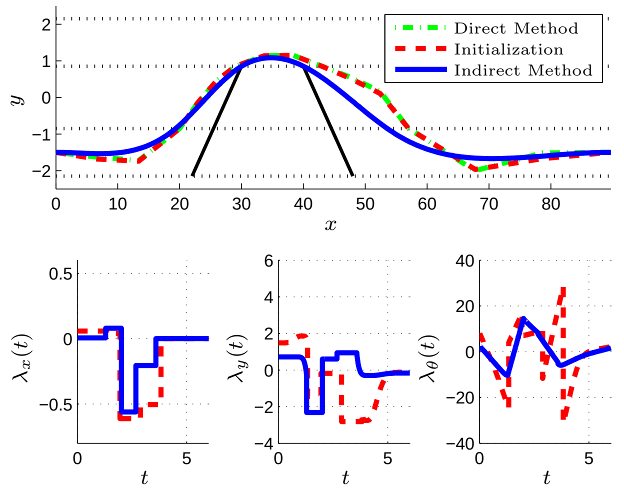


Fig. 4. Comparison of the trajectories found by concept 2 after optimizing with the direct method (green dotted line), after initializing the indirect method with the direct one (red dashed line), and after optimizing with the indirect one after the initialization (blue solid line).

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